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Collisions de points-vortex et confinement dans les domaines bornés

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Chapitre 1

Introduction

D'après le Larousse en ligne [57], le nom *fluide* désigne le "nom générique des liquides, des gaz et des plasmas, corps ayant en commun la propriété de pouvoir prendre n'importe quelle forme sous l'effet de forces aussi petites que l'on veut". Venant du latin *fluidus* qui signifie "qui coule", cette définition renferme bien l'intuition que l'on se fait d'un fluide. La *mécanique des fluides* est donc la "science qui étudie l'équilibre et le mouvement des fluides". Ce mémoire traite de mécanique des fluides : notre objectif sera donc de décrire des mouvements. En particulier, nous nous intéresserons aux *tourbillons* : "Mouvement rapides et circulaires".

Le chapitre 1 introduit les objets mathématiques que nous allons étudier. Le chapitre 2 présente et résume les différentes contributions de ce mémoire. Les chapitres 3, 4 et 5 sont constitués des articles scientifiques suivants :

Long time confinement of vorticity around a stable stationary point vortex in a bounded planar domain
Avec **Dragoş Iftimie**, Ann. Inst. H. Poincaré Anal. Non Linéaire, 2021, 38 (5), pp. 1461–1485.

Two-dimensional point vortex dynamics in bounded domains : global existence for almost every initial data
SIAM J. Math. Anal., 2021, 54 (1), pp. 79–113.

Hölder regularity of collapses of point vortices
Avec **Ludovic Godard-Cadillac**, arXiv :2111.14230.

1.1 Équations d'Euler

Les équations d'Euler sont les équations décrivant le mouvement d'un fluide non visqueux. On le supposera également incompressible et homogène.

Pour comprendre la physique du mouvement d'un fluide, nous allons considérer un *élément de fluide*. C'est un volume suffisamment petit de sorte que les grandeurs intensives soient presque constantes partout à l'intérieur de l'élément, mais suffisamment grand pour se comporter comme un milieu *continu*, et que tous les effets microscopiques soient moyennés. En physique, cet élément serait typiquement d'échelle *mésoscopique*.

Il existe deux façons traditionnelles de décrire un fluide. L'une d'elles, la description *Lagrangienne*, consiste à se concentrer sur un élément de fluide, et d'en étudier la trajectoire. La seconde description, dite *Eulerienne*, consiste à décrire le fluide par un champ de vitesse, défini en tout point comme la vitesse des éléments de fluide passant successivement en ce point. Nous verrons en section 1.1.1 en détail le point de vue Eulerien, puis en section 1.1.2 lorsque nous évoquerons le *flot* de l'équation, le point de vue Lagrangien.

En Annexe A.1.1 sont rappelées les définitions des différents opérateurs d'analyse vectorielle ainsi que les formules élémentaires de dérivation.

1.1.1 Approche physique

Puisque les équations fondamentales de la mécanique des fluides portent aujourd'hui son nom, intéressons nous à la façon dont Leonhard Euler, en 1757, décrivit en premier le mouvement d'un fluide parfait dans son mémoire [24]. Aidés des notes de lecture [28], décrivons son approche.

Les grandeurs qui caractérisent le fluide sont le champ de vitesse $\mathbf{u} : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}^3$, attribuant à chaque position (x, y, z) et chaque temps t la vitesse $\mathbf{u} = (u, v, w)$ du fluide, et sa densité $q : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$. En plus de celles-ci, Euler remarque que des grandeurs comme la température pourront être amenées à intervenir ; mais les considère alors comme des données du problème, et non des inconnues.

Considérons un élément de fluide parallélépipèdique de dimensions δx, δy, δz, qui est déplacé par le fluide pendant un temps infinitésimal δt. La première étape pour la description du mouvement est de calculer la variation du volume de l'élément de fluide après son déplacement. Bien que l'élément de fluide soit alors déformé, Euler remarque qu'au premier ordre d'approximation, l'élément de fluide reste un parallélépipède, dont les dimensions évoluent. Il remarque que sur chaque axe, la dimension de l'élément de fluide change en raison de la variation spatiale du champ de vitesse. Ceci est illustré en Figure 1.1. La dimension δx devient $\delta x(1 + \frac{\mathrm{d}u}{\mathrm{d}x}\delta t)$. De même, δy devient $\delta y(1 + \frac{\mathrm{d}v}{\mathrm{d}y}\delta t)$ et δz devient $\delta z(1 + \frac{\mathrm{d}w}{\mathrm{d}z}\delta t)$.

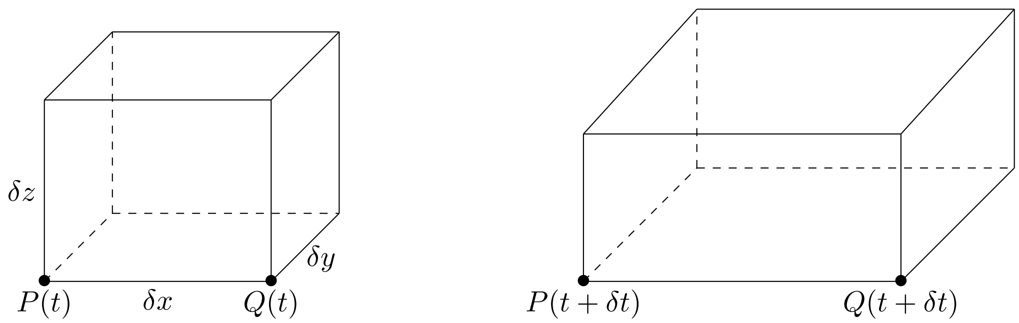


FIGURE 1.1 – L'évolution entre t et $t + \delta t$ de l'élément de fluide. Le point P évolue dans la direction (Ox) à la vitesse u tandis que le point Q évolue avec une vitesse $u + \delta x \frac{\mathrm{d}u}{\mathrm{d}t}$. Ainsi, la distance les séparant a évolué de $\delta x \frac{\mathrm{d}u}{\mathrm{d}t} \delta t$.

Il est important de distinguer les quantités infinitésimales écrites à l'aide du symbole δ, qui sont des quantités destinées à être choisies arbitrairement petites, et la fonction dérivée, écrite à l'aide de la fraction $\frac{\mathrm{d}\cdot}{\mathrm{d}\cdot}$. Ainsi, si f est une fonction de la variable t, alors $f(t + \delta t) = f(t) + \delta t \frac{\mathrm{d}f}{\mathrm{d}t}$, au premier ordre d'approximation.

Le volume de fluide reste le produit de ses dimensions sur chaque axe. Ainsi, on obtient que la variation de volume δV est au premier ordre

$$\delta V = \delta x \left(1 + \frac{\mathrm{d}u}{\mathrm{d}x}\delta t\right) \delta y \left(1 + \frac{\mathrm{d}v}{\mathrm{d}y}\delta t\right) \delta z \left(1 + \frac{\mathrm{d}w}{\mathrm{d}z}\delta t\right) - \delta x \delta y \delta z = \delta x \delta y \delta z \left(\delta t \frac{\mathrm{d}u}{\mathrm{d}x} + \delta t \frac{\mathrm{d}v}{\mathrm{d}y} + \delta t \frac{\mathrm{d}w}{\mathrm{d}z}\right).$$

La densité de l'élément de fluide est donnée par la fonction q, dépendant de x, y, z et t. Puisque l'élément de fluide s'est déplacé dans les trois directions d'espace ainsi que dans le temps, sa densité a alors varié d'une quantité

$$\delta q = \delta t \frac{\mathrm{d}q}{\mathrm{d}t} + \delta x \frac{\mathrm{d}q}{\mathrm{d}x} + \delta y \frac{\mathrm{d}q}{\mathrm{d}y} + \delta z \frac{\mathrm{d}q}{\mathrm{d}z}.$$

En choisissant $\delta x = u \delta t$, $\delta y = v \delta t$ et $\delta z = w \delta t$, on trouve,

$$\delta q = \delta t \frac{\mathrm{d}q}{\mathrm{d}t} + u \delta t \frac{\mathrm{d}q}{\mathrm{d}x} + v \delta t \frac{\mathrm{d}q}{\mathrm{d}y} + w \delta t \frac{\mathrm{d}q}{\mathrm{d}z}.$$

Enfin, utilisant intuitivement le fait que la masse de l'élément de fluide n'a pas changé au cours du mouvement, la densité est inversement proportionnelle au volume, de sorte que,

$$\frac{q+\delta q}{q} = \frac{V}{V+\delta V}.$$

En simplifiant et développant à l'ordre 1 on trouve

$$\frac{\delta q}{q} = -\frac{\delta V}{V},$$

c'est-à-dire

$$\delta t \frac{\mathrm{d}q}{\mathrm{d}t} + u\delta t \frac{\mathrm{d}q}{\mathrm{d}x} + v\delta t \frac{\mathrm{d}q}{\mathrm{d}y} + w\delta t \frac{\mathrm{d}q}{\mathrm{d}z} + q\left(\delta t \frac{\mathrm{d}u}{\mathrm{d}x} + \delta t \frac{\mathrm{d}v}{\mathrm{d}y} + \delta t \frac{\mathrm{d}w}{\mathrm{d}z}\right) = 0$$

ce qui donne en remarquant que $u\frac{\mathrm{d}q}{\mathrm{d}x} + q\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}(qu)}{\mathrm{d}x}$, et la même chose pour v et w, que

$$\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{\mathrm{d}(qu)}{\mathrm{d}x} + \frac{\mathrm{d}(qv)}{\mathrm{d}y} + \frac{\mathrm{d}(qw)}{\mathrm{d}z} = 0.$$

Cette équation, porte le nom d'équation de continuité, ou de conservation de la masse. Euler parle effectivement de son hypothèse de *continuité* du fluide ; une autre façon de le voir, est de constater que la supposition implicite que nous avons faite est que la masse est conservée pendant le déplacement de notre élément de fluide.

Établissons cette équation avec les notations modernes. Considérons un volume V et une quantité q transportée par un flot de vitesse u. Alors si la quantité q n'est ni produite, ni détruite, c'est-à-dire qu'elle est conservée, sa variation n'est due qu'à son flux à travers le bord du volume. C'est-à-dire

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V q(x,t)\mathrm{d}x = -\int_{\partial V} q(s,t)\mathbf{u}(s,t)\cdot n(s)\mathrm{d}s,$$

où n est la normale *sortante* à la surface ∂V. D'après le théorème de Green-Ostrogradski rappelé en Annexe A.1.1, on a donc

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V q(x,t)\mathrm{d}x + \int_V \nabla\cdot(q\mathbf{u})(x,t)\mathrm{d}x = 0.$$

En remarquant que ceci est vrai quel que soit le volume V, on obtient alors, pour tout x dans l'espace, et tout temps t,

$$\frac{\partial q}{\partial t}(x,t) + \nabla\cdot(q\mathbf{u})(x,t) = 0,$$

ce qui est bien l'équation obtenue précédemment.

Cette relation est une première contrainte pour l'évolution du fluide. La seconde contrainte est obtenue par le principe fondamental de la dynamique, ou seconde loi de Newton. Puisque le fluide est supposé parfait, la seule force interne dont Euler suppose l'existence est la pression p. Pour travailler en toute généralité, il introduit également une force *accélératrice* extérieure $F = (P, Q, R)$. Ici, accélératrice signifie qu'elle a la dimension d'une accélération. Pour décrire la force de pression, Euler commence par calculer la force *motrice* (bien dimensionnée comme une force) qu'elle engendre. La force de pression reçue par l'élément de fluide sur chaque axe est la différence des forces de pression reçues par les faces opposées, allant de la forte pression vers la faible pression. Ainsi le différentiel de pression entre les faces opposées est

$$\left(-\delta x\frac{\mathrm{d}p}{\mathrm{d}x}, -\delta y\frac{\mathrm{d}p}{\mathrm{d}y}, -\delta z\frac{\mathrm{d}p}{\mathrm{d}z}\right).$$

Puisque la force de pression est obtenue comme le produit de la pression (force surfacique) et de la surface sur laquelle elle s'applique, on obtient comme force de pression

$$-\delta x\delta y\delta z\left(\frac{\mathrm{d}p}{\mathrm{d}x}, \frac{\mathrm{d}p}{\mathrm{d}y}, \frac{\mathrm{d}p}{\mathrm{d}z}\right).$$

Pour obtenir la force accélératrice, il faut diviser par la masse de l'élément de fluide, valant $q\delta x\delta y\delta z$. La force accélératrice de pression s'écrit donc

$$-\frac{1}{q}\left(\frac{\mathrm{d}p}{\mathrm{d}x}, \frac{\mathrm{d}p}{\mathrm{d}y}, \frac{\mathrm{d}p}{\mathrm{d}z}\right).$$

Cherchons désormais à décrire l'accélération de l'élément de fluide. Celle-ci s'exprime comme la variation de vitesse $\delta\mathbf{u}$ de l'élément entre t et $t+\delta t$, divisé par le temps δt. Comme pour sa densité, puisque l'élément de fluide s'est déplacé dans les trois coordonnées d'espace et dans le temps, la variation de vitesse s'écrit donc

$$\begin{cases} \delta u = \delta t\dfrac{\mathrm{d}u}{\mathrm{d}t} + \delta x\dfrac{\mathrm{d}u}{\mathrm{d}x} + \delta y\dfrac{\mathrm{d}u}{\mathrm{d}y} + \delta z\dfrac{\mathrm{d}u}{\mathrm{d}z} \\ \delta v = \delta t\dfrac{\mathrm{d}v}{\mathrm{d}t} + \delta x\dfrac{\mathrm{d}v}{\mathrm{d}x} + \delta y\dfrac{\mathrm{d}v}{\mathrm{d}y} + \delta z\dfrac{\mathrm{d}v}{\mathrm{d}z} \\ \delta w = \delta t\dfrac{\mathrm{d}w}{\mathrm{d}t} + \delta x\dfrac{\mathrm{d}w}{\mathrm{d}x} + \delta y\dfrac{\mathrm{d}w}{\mathrm{d}y} + \delta z\dfrac{\mathrm{d}w}{\mathrm{d}z} \end{cases}$$

ce qui donne donc

$$\begin{cases} \delta u = \delta t\left(\dfrac{\mathrm{d}u}{\mathrm{d}t} + u\dfrac{\mathrm{d}u}{\mathrm{d}x} + v\dfrac{\mathrm{d}u}{\mathrm{d}y} + w\dfrac{\mathrm{d}u}{\mathrm{d}z}\right) \\ \delta v = \delta t\left(\dfrac{\mathrm{d}v}{\mathrm{d}t} + u\dfrac{\mathrm{d}v}{\mathrm{d}x} + v\dfrac{\mathrm{d}v}{\mathrm{d}y} + w\dfrac{\mathrm{d}v}{\mathrm{d}z}\right) \\ \delta w = \delta t\left(\dfrac{\mathrm{d}w}{\mathrm{d}t} + u\dfrac{\mathrm{d}w}{\mathrm{d}x} + v\dfrac{\mathrm{d}w}{\mathrm{d}y} + w\dfrac{\mathrm{d}w}{\mathrm{d}z}\right). \end{cases}$$

Le principe fondamental de la dynamique énonce alors que l'accélération de l'élément du fluide est égale à la somme des forces accélératrices qu'il subit. On obtient

$$\begin{cases} P - \dfrac{1}{q}\dfrac{\mathrm{d}p}{\mathrm{d}x} = \dfrac{\mathrm{d}u}{\mathrm{d}t} + u\dfrac{\mathrm{d}u}{\mathrm{d}x} + v\dfrac{\mathrm{d}u}{\mathrm{d}y} + w\dfrac{\mathrm{d}u}{\mathrm{d}z} \\ Q - \dfrac{1}{q}\dfrac{\mathrm{d}p}{\mathrm{d}y} = \dfrac{\mathrm{d}v}{\mathrm{d}t} + u\dfrac{\mathrm{d}v}{\mathrm{d}x} + v\dfrac{\mathrm{d}v}{\mathrm{d}y} + w\dfrac{\mathrm{d}v}{\mathrm{d}z} \\ R - \dfrac{1}{q}\dfrac{\mathrm{d}p}{\mathrm{d}z} = \dfrac{\mathrm{d}w}{\mathrm{d}t} + u\dfrac{\mathrm{d}w}{\mathrm{d}x} + v\dfrac{\mathrm{d}w}{\mathrm{d}y} + w\dfrac{\mathrm{d}w}{\mathrm{d}z}. \end{cases}$$

Avec les notations introduites en Annexe A.1.1, cette équation se réécrit

$$F - \frac{1}{q}\nabla p = \frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}.$$

Le système que nous avons obtenu jusqu'à présent comporte 4 équations, et 5 inconnues : (u, v, w, p, q). Euler précise que la 5ème équation qui clôt le système est une équation reliant p et q, et probablement d'autres *qualités* comme la température. Ne pas expliciter de telle formule n'est pas un problème pour la suite de l'étude qu'Euler fait. Aujourd'hui, on nomme cette équation manquante *équation d'état.* On l'obtient par exemple par un bilan d'énergie.

Pour les flots incompressibles, qui seront notre objet d'étude dans ce mémoire, Euler remarque que le volume de l'élément de fluide n'est plus modifié au cours de son mouvement. Ceci simplifie l'équation de continuité : en effet, on a cette fois que $\delta V = 0$. Ceci donne d'une part que

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}y} + \frac{\mathrm{d}w}{\mathrm{d}z} = 0,$$

et d'autre part que $\delta q = 0$, ce qui implique

$$\frac{\mathrm{d}q}{\mathrm{d}t} + u\frac{\mathrm{d}q}{\mathrm{d}x} + v\frac{\mathrm{d}q}{\mathrm{d}y} + w\frac{\mathrm{d}q}{\mathrm{d}z} = 0.$$

La densité est alors préservée durant le mouvement du fluide : elle est déplacée sans changer de valeur.

La condition d'incompressibilité se traduit directement, comme nous l'avons vu, par l'équation

$$\nabla \cdot \mathbf{u} = 0$$

qui, introduite dans l'équation de continuité, donne

$$\frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q = 0.$$

Enfin, si le fluide est homogène, c'est-à-dire que la densité est constante, que l'on peut choisir égale à 1, l'équation de continuité devient vide, et donc notre système de contraintes se réduit à

$$\begin{cases} \nabla \cdot \mathbf{u} = 0 \\ \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p. \end{cases}$$

Puisque la densité n'est plus une inconnue du problème, il est naturel que nous n'ayons plus que quatre équations. Ce sont celles-ci que nous allons étudier pour la majeure partie de ce mémoire. Elles sont appelées *équations d'Euler homogènes et incompressibles.*

1.1.2 Tourbillon

Le tourbillon, aussi appelé vorticité, a été introduit par Helmholtz dans son article [43], traduit en anglais dans [75]. Sa première remarque est la suivante : si autour d'un point le fluide est animé d'une vitesse angulaire $\frac{1}{2}(\xi, \eta, \zeta)$, alors

$$\begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}y} - \dfrac{\mathrm{d}v}{\mathrm{d}z} = \xi \\ \dfrac{\mathrm{d}u}{\mathrm{d}z} - \dfrac{\mathrm{d}w}{\mathrm{d}x} = \eta \\ \dfrac{\mathrm{d}v}{\mathrm{d}x} - \dfrac{\mathrm{d}u}{\mathrm{d}y} = \zeta \end{cases}$$

c'est à dire

$$\nabla \wedge \mathbf{u} = (\xi, \eta, \zeta).$$

Dans le contexte de la mécanique des fluides, on appelle vecteur de *tourbillon* du fluide le champ de vecteur $\mathbf{\Omega} := (\xi, \eta, \zeta) = \nabla \wedge \mathbf{u}$. Faisons maintenant une expérience de pensée. Plaçons en un point du fluide un solide infinitésimal. Si le tourbillon est nul, le fluide est dit irrationnel. Le solide suit alors le courant, mais garde au cours de son mouvement la même orientation. En revanche, si le tourbillon n'est pas nul, ce solide possède un axe de rotation instantané : la direction du vecteur $\mathbf{\Omega}$. Imaginons maintenant une courbe continue de solides infinitésimaux immergée dans un fluide, de sorte qu'en tout point de cette courbe, l'axe de rotation instantané du solide en ce point soit tangent à la courbe. On obtient alors une courbe de solide qui tourne sur elle-même.

FIGURE 1.2 – Filament tourbillon. À gauche, le modèle mathématiques, à droite, une illustration de ce qu'il modélise.
Crédit image : https ://en.wikipedia.org/wiki/File :Dszpics1.jpg. Appartient au domaine public.

Un cas particulièrement intéressant est le cas où le tourbillon se concentre autour d'une telle courbe avec une intensité fixée : notons Γ la courbe et $\mathbf{t}$ son vecteur tangent pour une orientation choisie. On s'intéresse donc au cas où

$$\mathbf{\Omega} \approx \mathbf{t}\, \delta_\Gamma,$$

où δ désigne la mesure de Dirac. L'illustration est donnée en Figure 1.2 : le fluide tourbillonne autour de cette courbe. C'est par exemple ce que l'on appelle communément une tornade. On appelle la construction singulière un *filament tourbillon*. Cette courbe peut être fermée : on l'appelle alors *anneau tourbillon*. C'est encore une fois un phénomène communément observable, puisque ceci décrit par exemple les ronds de fumée que peuvent former les fumeurs aguerris. On pourra trouver des détails sur ces anneaux tourbillons dans [64].

Pour obtenir une description similaire en dimension 2, on peut par exemple supposer que le champ de vitesse du fluide est invariant dans la direction (Oz), et en tout point parallèle au plan (Oxy). Le tourbillon vaut alors

$$\mathbf{\Omega} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{\mathrm{d}v}{\mathrm{d}x} - \dfrac{\mathrm{d}u}{\mathrm{d}y}. \end{pmatrix}$$

On pose donc $\omega = \frac{\mathrm{d}v}{\mathrm{d}x} - \frac{\mathrm{d}u}{\mathrm{d}y}$, qu'on appellera également tourbillon du fluide, pour un fluide 2-dimensionnel. Cette fois, la structure singulière naturelle est un *point-vortex*, dont nous détaillerons grandement les propriétés en Section 1.2. Il s'agit en termes commun d'un tourbillon.

À partir de l'équation du mouvement d'Euler, Helmholtz obtient l'équation d'évolution suivante :

$$\begin{cases} \frac{\mathrm{d}\xi}{\mathrm{d}t} + u\frac{\mathrm{d}\xi}{\mathrm{d}x} + v\frac{\mathrm{d}\xi}{\mathrm{d}y} + w\frac{\mathrm{d}\xi}{\mathrm{d}z} = \xi\frac{\mathrm{d}u}{\mathrm{d}x} + \eta\frac{\mathrm{d}u}{\mathrm{d}y} + \zeta\frac{\mathrm{d}u}{\mathrm{d}z} \\ \frac{\mathrm{d}\eta}{\mathrm{d}t} + u\frac{\mathrm{d}\eta}{\mathrm{d}x} + v\frac{\mathrm{d}\eta}{\mathrm{d}y} + w\frac{\mathrm{d}\eta}{\mathrm{d}z} = \xi\frac{\mathrm{d}v}{\mathrm{d}x} + \eta\frac{\mathrm{d}v}{\mathrm{d}y} + \zeta\frac{\mathrm{d}v}{\mathrm{d}z} \\ \frac{\mathrm{d}\zeta}{\mathrm{d}t} + u\frac{\mathrm{d}\zeta}{\mathrm{d}x} + v\frac{\mathrm{d}\zeta}{\mathrm{d}y} + w\frac{\mathrm{d}\zeta}{\mathrm{d}z} = \xi\frac{\mathrm{d}w}{\mathrm{d}x} + \eta\frac{\mathrm{d}w}{\mathrm{d}y} + \zeta\frac{\mathrm{d}w}{\mathrm{d}z} \end{cases}$$

soit

$$\frac{\partial\mathbf{\Omega}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{\Omega} = \mathbf{\Omega}\cdot\nabla\mathbf{u}.$$

Cette équation décrit l'évolution du tourbillon dans le fluide. En dimension 2, elle se simplifie, et l'on obtient

$$\frac{\partial\omega}{\partial t} + \mathbf{u}\cdot\nabla\omega = 0. \tag{1.1}$$

Cette équation a la forme d'une équation de transport. La non-linéarité du problème est cachée dans la dépendance de u en ω, que l'on détaillera en Section 1.1.3. Néanmoins, cette structure d'équation de transport a des conséquences. La plus importante est la conservation du tourbillon le long des *trajectoires*.

Si l'on note par $\phi(x,t)$ la position d'un élément de fluide issu du point x après un temps t, on a très naturellement les équations suivantes :

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\phi(x,t) = \mathbf{u}(\phi(x,t),t), \\ \phi(x,0) = x. \end{cases}$$

Les fonctions $t \mapsto \phi(x,t)$ sont appelées *trajectoires* des éléments de fluides, la fonction $(x,t) \mapsto \phi(x,t)$ est appelée *flot* de l'équation. D'après le théorème de Cauchy-Lipschitz, si $\mathbf{u}$ est assez régulière, alors ϕ aussi. En particulier, si $\mathbf{u}$ est C^1, alors $x \mapsto \phi(x,t)$ est un difféomorphisme de $\mathbb{R}^2$.

À l'aide de l'équation (1.1), on peut alors calculer la variation du tourbillon le long des trajectoires :

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega(\phi(x,t),t) = \frac{\partial\omega}{\partial t}(\phi(x,t),t) + \frac{\mathrm{d}}{\mathrm{d}t}\phi(x,t)\cdot\nabla\omega(\phi(x,t),t) = 0,$$

et donc pour tout $t \geq 0$ et tout x,

$$\omega(\phi(x,t),t) = \omega_0(x). \tag{1.2}$$

On en déduit par exemple la conservation des normes L^p : soit $p \in [1,\infty]$, alors

$$\|\omega\|_{L^p} = \|\omega_0\|_{L^p}.$$

Ceci clôt nos considérations historiques sur l'établissement des équations de la mécanique des fluides. Pour le reste de ce mémoire, nous adoptons désormais les notations mathématiques usuelles. En dimension 2, le champ de vitesse est noté $u = (u_1, u_2)$ et son tourbillon $\omega = \partial_1 u_2 - \partial_2 u_1$.

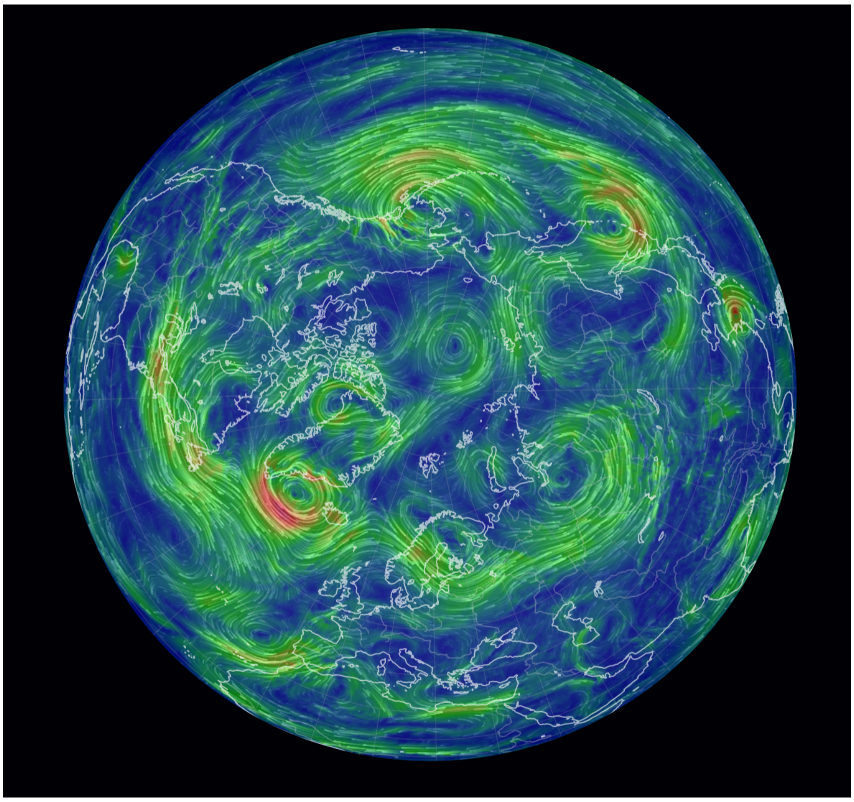


FIGURE 1.3 – Tourbillon bidimensionnel : coupe isobare des courants atmosphériques terrestres.
Crédit image : https ://earth.nullschool.net. Avec l'autorisation de son propriétaire. Paramètres donnés en Annexe A.4.

1.1.3 Loi de Biot Savart

À partir du champ de vitesse, nous avons introduit le tourbillon. L'objectif de cette section est d'obtenir une formule inverse donnant u en fonction de ω, c'est à dire de résoudre le problème :

$$\begin{cases} \operatorname{curl} u = \omega & \text{sur } \Omega \\ \nabla \cdot u = 0 & \text{sur } \Omega \\ u \cdot n = 0 & \text{sur } \partial\Omega. \end{cases} \tag{1.3}$$

La dernière équation est une contrainte au bord : il est naturel d'imposer que le fluide ne puisse pas quitter le domaine. Notons bien qu'il ne s'agit pas de choisir $u = 0$ au bord. Le fluide peut avoir une vitesse non nulle au bord, mais alors tangente à celui-ci, de sorte à ne pas quitter le domaine.

Ce problème va dépendre fortement du domaine dans lequel évolue le fluide. L'idée générale est la suivante : puisque u est à divergence nulle, dans les domaines simplement connexes, en utilisant le Lemme de Poincaré à $u^\perp$, on obtient que pour chaque temps t il existe une fonction de courant $\psi(\cdot, t) : \mathbb{R}^2 \mapsto \mathbb{R}^2$ telle que

$$u(x,t) = \nabla^\perp \psi(x,t) = \begin{pmatrix} -\partial_2 \psi(x,t) \\ \partial_1 \psi(x,t) \end{pmatrix}.$$

Calculons donc la relation entre ψ et ω. On a :

$$\begin{aligned} \omega(x,t) &= \partial_1 u_2 - \partial_2 u_1 \\ &= \partial_1^2 \psi(x,t) + \partial_2^2 \psi(x,t) \\ &= \Delta \psi(x,t). \end{aligned}$$

Ainsi, si l'on sait inverser le Laplacien, alors on peut écrire $\psi(x,t) = \Delta^{-1}\omega(x,t)$, et donc

$$u(x,t) = \nabla^{\perp}\Delta^{-1}\omega(x,t).$$

On a donc une formule exprimant u en fonction de ω, dans les domaines simplement connexes et à condition de savoir inverser le Laplacien. Nous allons étudier le cas du plan entier, puis des domaines bornés simplement connexes, et enfin des domaines non simplement connexes.

Pour cela, rappelons que la fonction de Green G_Ω du Laplacien d'un domaine Ω, satisfait (se référer par exemple à [25]) :

$$\forall x \neq y \in \mathbb{R}^2, \quad G_\Omega(x,y) = G_\Omega(y,x),$$

l'égalité au sens des distributions :

$$\forall y \in \mathbb{R}^2, \quad \Delta G_\Omega(\cdot, y) = \delta_y$$

où δ_y désigne la masse de Dirac au point y, et la condition de bord

$$G_\Omega(x,y) \underset{x\to\partial\Omega}{\longrightarrow} 0.$$

Dans le plan entier

Il est bien connu que la fonction de Green du plan est donnée par

$$G_{\mathbb{R}^2}(x,y) = \frac{1}{2\pi}\ln|x-y|,$$

et que (voir [30]) si $\omega \in L^1 \cap L^\infty$, le problème

$$\omega = \Delta\psi \tag{1.4}$$

a une solution

$$\psi(x) = \int G_{\mathbb{R}^2}(x,y)\omega(y)\mathrm{d}y = \int \frac{1}{2\pi}\ln|x-y|\omega(y)\mathrm{d}y. \tag{1.5}$$

L'ensemble des distributions tempérées solutions du problème (1.4) est alors l'ensemble des fonctions $\psi + P$, où P est un polynôme harmonique.

Puisque l'on cherche u comme le gradient orthogonal de cette solution $\psi + P$, en imposant $u \underset{|x|\to+\infty}{\longrightarrow} 0$, on obtient que P doit être constante, et donc que u est uniquement définie. En imposant à u cette condition à l'infini, on s'assure donc de l'unicité de la solution du problème (1.3). La solution est donnée par la formule, dite *loi de Biot-Savart*, suivante :

$$u(x,t) = \int \frac{(x-y)^\perp}{2\pi|x-y|^2}\omega(y,t)\mathrm{d}y. \tag{1.6}$$

Dans un domaine borné simplement connexe

Si le domaine d'étude est un domaine borné simplement connexe, l'étude est similaire, mais la fonction de Green est moins explicite en général. Puisque $u \cdot n = 0$, la fonction de courant ψ est donc constante au bord. Puisqu'elle est définie à une constante près, on peut choisir la constante de sorte que $\psi = 0$ au bord. Ceci permet une nouvelle fois d'écrire

$$u(x,t) = \int \nabla_x^\perp G_\Omega(x,y)\omega(y,t)\mathrm{d}y.$$

Il est possible de développer un peu plus cette formule. En effet, si l'on calcule, au sens des distributions et à $y \in \Omega$ fixé, le Laplacien de la fonction $x \mapsto G_\Omega(x,y) - G_{\mathbb{R}^2}(x,y)$, on obtient 0.

La fonction $x \mapsto G_\Omega(x,y) - G_{\mathbb{R}^2}(x,y)$ est donc harmonique, et en particulier, elle est régulière. Notons $x \mapsto \gamma_\Omega(x,y)$ cette fonction. On a donc

$$G_\Omega(x,y) = \frac{1}{2\pi} \ln|x-y| + \gamma_\Omega(x,y). \tag{1.7}$$

Ainsi, les fonctions de Green de tous les domaines ont la même partie singulière. Ceci donne pour la loi de Biot-Savart :

$$u(x,t) = \int \left(\frac{(x-y)^\perp}{2\pi|x-y|^2} + \nabla_x^\perp \gamma_\Omega(x,y) \right) \omega(y,t)\mathrm{d}y. \tag{1.8}$$

Dans un domaine borné non simplement connexe

Si le domaine n'est plus simplement connexe, alors notre méthode ne fonctionne plus : on ne peut plus appliquer le Lemme de Poincaré à u.

Tout d'abord, on peut vérifier que la fonction suivante

$$x \mapsto \int \nabla_x^\perp G_\Omega(x,y)\omega(y)\mathrm{d}y$$

est toujours une solution particulière du problème (1.3). Puisque ce problème est linéaire, il ne reste alors qu'à étudier le problème homogène

$$\begin{cases} \mathrm{curl}\, u = 0 & \text{sur } \Omega \\ \nabla \cdot u = 0 & \text{sur } \Omega \\ u \cdot n = 0 & \text{sur } \partial\Omega. \end{cases} \tag{1.9}$$

Soit Γ_0 la frontière extérieure de Ω, et soient $\Gamma_1, \ldots, \Gamma_m$ les bords intérieurs, c'est à dire englobant des "trous", comme décrit en Figure 1.4. Pour chaque bord intérieur Γ_j, $j \in [\![1,m]\!]$

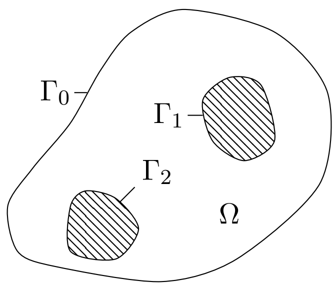


FIGURE 1.4 – Exemple de domaine Ω non simplement connexe.

notons $\xi_j(t)$ la circulation de u sur Γ_j donnée par

$$\xi_j(t) = \int_{\Gamma_j} u(s,t) \cdot (-n^\perp)\mathrm{d}s,$$

où nous rappelons que n est la normale sortante unitaire au domaine. La présence du signe $-$ provient du fait que l'on oriente usuellement les circulations dans le sens trigonométrique, et qu'à l'intérieur du domaine, le vecteur tangent à Γ_j pour $j \neq 0$ orienté dans le sens trigonométrique est $-n^\perp$.

D'après la théorie de Hodge, on a l'existence d'une base de solutions β_j, $j \in [\![1,m]\!]$, du problème homogène 1.9 vérifiant

$$\int_{\Gamma_\ell} \beta_j \cdot (-n^\perp)\mathrm{d}s = \delta_{j,\ell} \quad \text{pour } l \in [\![1,m]\!].$$

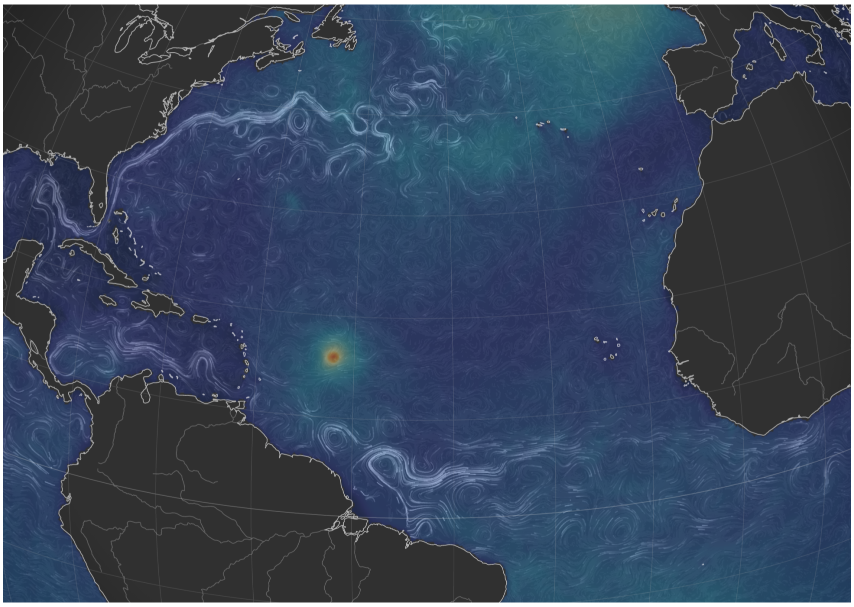


FIGURE 1.5 – Tourbillon dans un domaine borné non simplement connexe : courants océaniques.
Crédit image : https ://earth.nullschool.net. Avec l'autorisation de son propriétaire. Paramètres donnés en Annexe A.4.

Les fonctions β_j sont appelées *champs harmoniques*. Ainsi, on sait que u s'écrit de manière unique à l'aide de fonctions $t \mapsto c_j(t)$, telles que

$$u(x,t) = \int_\Omega \nabla_x^\perp G_\Omega(x,y)\omega(y,t)\mathrm{d}y + \sum_{j=1}^m c_j(t)\beta_j(x). \tag{1.10}$$

C'est notre loi de Biot-Savart pour les domaines bornés non simplement connexes. Précisons la valeur des fonctions c_j.

Tout d'abord, d'après le théorème de Kelvin [77], les circulations ξ_j de u sont constantes en temps, et donc prescrites par la donnée initiale. En effet, comme présenté dans [27], on a d'une part par un bref calcul que

$$u \cdot \nabla u = \frac{1}{2}\nabla|u|^2 + \omega u^\perp$$

et donc

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi_j(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_j} u\cdot(-n^\perp) = -\int_{\Gamma_j}\nabla\Big(\frac{|u|^2}{2}+p\Big)\cdot(-n^\perp)\mathrm{d}s - \int_{\Gamma_j}\omega u^\perp\cdot(-n^\perp)\mathrm{d}s = 0.$$

Le premier terme du membre de droite de la loi de Biot-Savart (1.10) n'étant pas de circulation nulle sur chaque bord, un calcul détaillé en Annexe A.2.2 donne alors que les c_j s'écrivent

$$c_j(t) = \int_\Omega \omega(x,t)w_j(x)\mathrm{d}x + \xi_j, \tag{1.11}$$

où les fonctions w_j sont appelées *mesures harmoniques* et définies pour $j \in [\![1,m]\!]$ comme l'unique solution du problème

$$\begin{cases} \Delta w_j = 0 & \text{sur } \Omega \\ w_j = \delta_{j,\ell} & \text{sur } \Gamma_\ell,\ \ 0 \le \ell \le m. \end{cases}$$

1.1.4 Existence et unicité des solutions

Commençons par faire le bilan des sections précédentes. Supposons que u a une régularité suffisante, ce qui sera toujours une supposition raisonnable dans le cadre de ce mémoire. Nous avons établi en section 1.1.1 les équations d'Euler incompressibles

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p & \text{sur }]0, +\infty[\times\Omega \\ \nabla \cdot u = 0 & \text{sur } [0, +\infty[\times\Omega. \end{cases}$$

Nous avons évoqué en section 1.1.3 qu'il est pertinent d'adjoindre la condition de bord $u \cdot n = 0$ sur $\partial\Omega$ lorsque Ω a un bord ainsi que $u \underset{|x|\to+\infty}{\longrightarrow} 0$ si Ω est non borné. Nous prescrivons aussi la donnée initiale u_0 à divergence nulle. On obtient donc dans le plan entier

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p & \text{sur }]0, +\infty[\times\mathbb{R}^2 \\ \nabla \cdot u = 0 & \text{sur } [0, +\infty[\times\mathbb{R}^2 \\ u(\cdot, 0) = u_0 & \text{sur } \mathbb{R}^2 \\ u \underset{|x|\to+\infty}{\longrightarrow} 0 & \text{sur } [0, +\infty[, \end{cases} \tag{1.12}$$

et dans un domaine borné

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p & \text{sur }]0, +\infty[\times\Omega \\ \nabla \cdot u = 0 & \text{sur } [0, +\infty[\times\Omega \\ u(\cdot, 0) = u_0 & \text{sur } \Omega \\ u \cdot n = 0 & \text{sur } [0, +\infty[\times\partial\Omega. \end{cases} \tag{1.13}$$

Insistons sur le fait que si u est connue, alors ∇p est déterminée et donc il n'est pas nécessaire de prescrire de condition initiale sur la pression p.

Reformulons ces systèmes en tourbillon. Nous avons obtenu en Section 1.1.2 une équation mécanique pour le tourbillon, et en Section 1.1.3 la loi de Biot-Savart, dépendant du domaine. Ceci nous donne le système

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, & \text{sur } \Omega \times \mathbb{R}_+ \\ u = \mathrm{BS}(\omega), & \text{obtenue par (1.6), (1.8) ou (1.10)} \\ \omega(\cdot, 0) = \omega_0, & \text{sur } \Omega. \end{cases} \tag{1.14}$$

Lorsque le domaine Ω n'est pas simplement connexe, les circulations ξ_j (constantes en temps) doivent également être prescrites pour définir correctement la loi de Biot-Savart (1.10), où les c_j sont données par la relation (1.11). On suppose désormais que ces circulations sont fixées.

Lorsque les conditions de régularité sont suffisantes, les formulations en vitesse et en vorticité sont équivalentes, et vérifient l'existence et l'unicité de solutions. Le premier théorème d'existence de solution remonte à Wolibner [80]. Citons le théorème d'existence de Yudovitch pour le problème en tourbillon. On considère Ω étant soit $\mathbb{R}^2$, soit un domaine borné de $\mathbb{R}^2$.

Théorème 1.1.1 (Yudovitch, 1963, [81]). *Soit $\omega_0 \in L^1(\Omega) \cap L^\infty(\Omega)$. Alors il existe une unique solution $\omega \in L^\infty(\mathbb{R}_+, L^1 \cap L^\infty)$ du problème* (1.14).

Dans le cadre de ce mémoire, nous supposerons toujours les données initiales suffisamment régulières pour que ces théorèmes puissent s'appliquer, de sorte que nos solutions sont toujours définies globalement en temps et uniques. La régularité des solutions obtenues est également suffisante pour tout ce que nous souhaitons étudier.

1.2 Système point-vortex

Comme nous l'avons déjà mentionné en Section 1.1.2, le tourbillon décrit la vitesse de rotation instantanée du fluide en chaque point. Posons

$$\forall x \neq 0, \quad K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}. \tag{1.15}$$

Alors la loi de Biot-Savart dans le plan (1.6) s'écrit comme le produit de convolution

$$u(x,t) = K * \omega(\cdot, t)$$

On peut donc interpréter K comme une solution fondamentale du problème (1.3) donnant u en fonction de ω, au sens où c'est le champ u produit par une masse de Dirac placée en 0.

On appelle un tourbillon de la forme $a\delta_z$ un *point-vortex* placé en z et d'*intensité* a. Notons que l'on **ne sait pas** donner de sens à une solution des équations d'Euler issue d'un tel tourbillon initial. En effet, il faut pour donner du sens au terme $u \cdot \nabla u$ que $u \in L^2_{\text{loc}}$. Or, si ω est une somme de masses de Dirac, à cause de cette singularité de K qui n'est pas L^2_{loc}, u ne l'est pas non plus.

L'objectif de cette section est de construire une fonction $t \mapsto \omega(\cdot, t)$, satisfaisant $\omega(\cdot, 0) = \sum_{i=1}^N a_i \delta_{z_i}$, *compatible* avec les équations d'Euler dans un sens qui ne pourra être que formel et non rigoureux.

1.2.1 Dynamique du système point-vortex

Soit $N \in \mathbb{N}$, $(z_i)_{1 \leq i \leq N} \in (\mathbb{R}^2)^N$ et $(a_i)_{1 \leq i \leq N} \in \mathbb{R}^N$. On pose

$$\omega_0 = \sum_{i=1}^N a_i \delta_{z_i}.$$

Rappelons que dans l'équation d'Euler, le tourbillon est conservé le long des trajectoires, voir l'équation (1.2). Il est donc raisonnable d'imposer, pour rester compatible avec les équations d'Euler, à ce que le tourbillon reste une somme de masse de Dirac :

$$\omega(x,t) = \sum_{i=1}^N a_i \delta_{z_i(t)}$$

où les points $z_i(t)$ suivent le flot de l'équation, c'est-à-dire $z_i(t) = \phi(z_i, t)$, avec $z_i(0) = z_i$. Ceci donne en particulier :

$$\frac{\mathrm{d}}{\mathrm{d}t} z_i(t) = \frac{\mathrm{d}}{\mathrm{d}t} \phi(z_i(t), t) = u(z_i(t), t).$$

Bien entendu, ici, le flot ϕ est mal défini, mais insistons encore une fois que ce que nous faisons *n'est pas* rigoureux. Cette dernière équation est très naturelle : puisque dans l'équation d'Euler, le tourbillon est transporté par le champ de vitesse u, alors les points-vortex, qui sont du tourbillon (singulier), le sont aussi.

La loi de Biot-Savart dans le plan (1.6) donne :

$$u(x,t) = \frac{1}{2\pi} \sum_{i=1}^N a_i \frac{(x - z_i(t))^\perp}{|x - z_i(t)|^2},$$

et donc on obtient en évaluant en z_i :

$$\frac{\mathrm{d}}{\mathrm{d}t} z_i(t) = \frac{1}{2\pi} \sum_{j=1}^N a_j \frac{(z_i(t) - z_j(t))^\perp}{|z_i(t) - z_j(t)|^2}. \tag{1.16}$$

Notre choix est mis en défaut à ce niveau, puisque cette formule n'est pas permise : le terme $j = i$ de la somme est $\frac{(z_i(t)-z_i(t))^\perp}{|z_i(t)-z_i(t)|^2}$ qui n'a pas de sens. Ce terme correspond à l'influence du point-vortex sur lui-même. Ce problème correspond au fait que le champ K définit en (1.15), qui exprime le champ de vitesse engendré par un point-vortex, est singulier en 0, c'est-à-dire là où le point-vortex auquel il correspond se trouve.

Le champ K est antisymétrique et orthoradial, il est raisonnable de se contenter de retirer le terme singulier de la somme dans l'équation (1.16). Ceci revient à prolonger K par $K(0) = 0$, ou encore à supposer qu'un point-vortex n'a pas d'influence sur lui-même. En procédant ainsi, on obtient désormais :

$$\forall i \in [\![1, N]\!], \quad \frac{\mathrm{d}}{\mathrm{d}t} z_i(t) = \frac{1}{2\pi} \sum_{\substack{1 \le j \le N \\ j \ne i}} a_j \frac{(z_i(t) - z_j(t))^\perp}{|z_i(t) - z_j(t)|^2}. \tag{1.17}$$

On appelle ce système d'équations *système point-vortex*. C'est un système différentiel ordinaire du premier ordre. Il est également non linéaire et autonome. En posant $Z(t) = (z_1(t), \dots, z_N(t))$, il s'écrit sous la forme

$$\frac{\mathrm{d}}{\mathrm{d}t} Z(t) = F(Z(t)) \tag{1.18}$$

où F est de classe C^∞ mais seulement sur l'ensemble des *données initiales admissibles* :

$$\mathcal{A} = \{Z \in (\mathbb{R}^2)^N, \ \ \forall i \ne j, \ \ z_i \ne z_j\}. \tag{1.19}$$

En particulier, on en conclut via le théorème de Cauchy-Lipschitz l'existence et l'unicité d'une solution maximale pour toute donnée initiale admissible. Ces solutions ne sont pas nécessairement globales en temps. Mais par le critère d'explosion en temps fini, on sait que si $T^* < \infty$ est le temps de vie maximal d'une solution, c'est que $F(t)$ quitte tout compact K de $(\mathbb{R}^2)^N$ lorsque $t \to T^*$. On en déduit que les solutions qui ne sont pas globales en temps doivent conduire à une *collision*, c'est-à-dire que

$$\exists i \ne j, \quad \liminf_{t \to T^*} |z_i(t) - z_j(t)| = 0.$$

Nous détaillerons ce problème en Section 2.2.

Nous avons établi le système point-vortex (1.17) dans le plan. Rappelons que bien que nous ayons établi ce système à l'aide de propriétés des équations d'Euler, nous n'avons pas encore fait de lien rigoureux avec celles-ci. Ce sera le but de la Section 1.2.4. Néanmoins, nous avons utilisé pour l'établir la loi de Biot-Savart dans le plan. Il est donc naturel d'étendre notre étude au cas des domaines bornés simplement connexes à l'aide de loi de Biot-Savart (1.8). On obtient de façon formelle que

$$\frac{\mathrm{d}}{\mathrm{d}t} z_i(t) = \sum_{j=1}^{N} a_j \nabla_x^\perp G_\Omega(z_i(t), z_j(t)).$$

Une nouvelle fois, le terme $G_\Omega(z_i(t), z_i(t))$ est singulier. Rappelons qu'en Section 1.1.3, nous avons établi la décomposition (1.7), rappelant que la singularité de G_Ω est la même que celle de $G_{\mathbb{R}^2}$. On traite donc ce terme de la même manière que précédemment : en retirant le terme singulier, à savoir $G_{\mathbb{R}^2}(z_i(t), z_i(t))$. On obtient donc cette fois :

$$\forall i \in [\![1, N]\!], \quad \frac{\mathrm{d}}{\mathrm{d}t} z_i(t) = \sum_{\substack{1 \le j \le N \\ j \ne i}} a_j \nabla_x^\perp G_\Omega(z_i(t), z_j(t)) + a_i \nabla_x^\perp \gamma_\Omega(z_i(t), z_i(t)). \tag{1.20}$$

Insistons sur la présence du dernier terme, qui n'est pas singulier. Dans le plan, l'influence d'un point-vortex sur lui-même, est nulle. Dans un domaine borné, cette influence n'est plus nulle, mais est donnée par $a_i \nabla_x^\perp \gamma_\Omega(z_i(t), z_i(t))$. La décomposition (1.7) traduit le fait que l'influence d'un point-vortex se décompose comme son influence directe, c'est-à-dire celle qu'il aurait dans le plan, plus un terme correcteur dû à la présence du bord. Autrement dit, le terme $a_i\nabla_x^\perp\gamma_\Omega(z_i(t),z_i(t))$ correspond à l'influence du point-vortex sur lui-même à travers le bord. On appelle fonction de *Robin* la fonction définie par

$$\widetilde{\gamma}_\Omega(x)=\gamma_\Omega(x,x). \tag{1.21}$$

On peut alors remplacer dans (1.20) le terme $\nabla_x^\perp\gamma_\Omega(z_i(t),z_i(t))$ par $\frac{1}{2}\nabla^\perp\widetilde{\gamma}_\Omega(z_i(t))$.

L'ensemble des conditions admissibles est

$$\mathcal{A}_\Omega=\{Z\in\Omega^N,\ \ \forall i\neq j,\ \ z_i\neq z_j\}. \tag{1.22}$$

Mais en écrivant à nouveau le système (1.20) sous la forme (1.18), on obtient cette fois qu'au voisinage du temps maximal fini d'existence d'une solution, on a soit

$$\exists i\neq j,\quad \liminf_{t\to T^*}|z_i(t)-z_j(t)|=0,$$

soit

$$\exists i\in[\![1,N]\!],\quad \liminf_{t\to T^*} d(z_i(t),\partial\Omega)=0,$$

puisqu'on a la propriété suivante, que l'on détaillera en section (1.2.3) : $|\nabla\widetilde{\gamma}(z)|\underset{z\to\partial\Omega}{\longrightarrow}+\infty$. On a donc, en plus d'une collision possible entre deux points-vortex, la possibilité qu'un point-vortex collisionne avec le bord du domaine.

Enfin, si le domaine n'est pas simplement connexe, la loi de Biot-Savart se complique encore, mais les nouveaux termes qu'il faut ajouter sont réguliers sur $\overline{\Omega}$ entier, et donc les conditions d'explosion en temps fini ne changent pas du cas simplement connexe. On obtient cette fois comme dynamique du système point-vortex :

$$\forall i\in[\![1,N]\!],\ \ \frac{\mathrm{d}}{\mathrm{d}t}z_i(t)=\sum_{\substack{1\le j\le N\\ j\neq i}}a_j\nabla_x^\perp G_\Omega(z_i(t),z_j(t))+\frac{1}{2}a_i\nabla^\perp\widetilde{\gamma}_\Omega(z_i(t))+\sum_{j=0}^{m}c_j(t)\beta_j(z_i(t)). \tag{1.23}$$

1.2.2 Généralisations, α-modèles

Le système point-vortex apparaît dans divers problèmes variés et pas seulement dans le domaine des fluides incompressibles. Citons par exemple le cas des équations de Gross-Pitaevskii, ou équations de Schrödinger-Ginzburg-Landau

$$i\partial_t u=\Delta u+\frac{1}{\varepsilon^2}u(1-|u|^2)$$

où $u:\mathbb{C}\to\mathbb{C}$. Ces équations traitent de la dynamique d'un gaz à l'état de condensat de Bose-Einstein. Alors lorsque ε tend vers 0, l'énergie

$$E_\varepsilon(u)=\frac{1}{2}\int|\nabla u|^2+\frac{1}{2\varepsilon^2}(1-|u|^2)^2 \tag{1.24}$$

fait apparaître des singularités dont le mouvement est exactement le système point-vortex (1.17). Ceci a été prouvé dans [19].

Un autre exemple beaucoup plus proche de l'étude que nous avons faite sont les équations *surface quasi-geostrophic*, que l'on abrégera en SQG. Liées aux équations d'Euler, elles peuvent s'écrire sous la formulation suivante

$$\begin{cases}\partial_t\omega+u\cdot\nabla\omega=0,\\ u=-\nabla^\perp(-\Delta)^{-s}\,\omega,\end{cases} \tag{SQG}$$

où $s \in]0,1[$. L'inversion du Laplacien fractionnaire, que l'on peut effectuer dans le plan entier donne la loi de Biot-Savart suivante :

$$u(x,t) = \int C_s \frac{(x-y)^\perp}{|x-y|^{4-2s}} \omega(y,t) \mathrm{d}y \tag{1.25}$$

avec

$$C_s = \frac{\Gamma(1-s)^2}{2^{2s-1}\,\pi\,\Gamma(s)}$$

où Γ est la fonction Gamma d'Euler. Ainsi, le même raisonnement qu'en Section 1.2.1 permet de définir le système point-vortex pour les équations SQG de la façon suivante :

$$\forall i \in [\![1,N]\!], \quad \frac{\mathrm{d}}{\mathrm{d}t} z_i(t) = C_s \sum_{\substack{1 \leq j \leq N \\ j \neq i}} a_j \frac{(z_i(t) - z_j(t))^\perp}{|z_i(t) - z_j(t)|^{4-2s}}. \tag{1.26}$$

Par un changement d'échelle de temps, on peut toujours se ramener au cas où la constante devant la somme vaut 1. C'est pourquoi on est amenés à définir une classe plus générale de systèmes point-vortex, que l'on nommera α-modèles, définis par :

$$\forall i \in [\![1,N]\!], \quad \frac{\mathrm{d}}{\mathrm{d}t} z_i(t) = \sum_{\substack{1 \leq j \leq N \\ j \neq i}} a_j \frac{(z_i(t) - z_j(t))^\perp}{|z_i(t) - z_j(t)|^{\alpha+1}}. \tag{1.27}$$

En particulier, le cas $\alpha = 1$ redonne, à constante près, le système point-vortex des équations d'Euler dans le plan (1.17).

1.2.3 Quelques propriétés

Une étude détaillée de la dynamique du système point-vortex est faite dans [65]. Détaillons quelques propriétés.

Système Hamiltonien

Le système point-vortex fait partie de la catégorie des systèmes *Hamiltoniens*. Ces systèmes sont généralement décrits sous la forme suivante. Soit $\mathcal{H}$ une fonctionnelle de trois variables p, q et t. Alors le système $t \mapsto (p(t), q(t))$ est dit Hamiltonien s'il vérifie les équations d'Hamilton suivante :

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t}(t) = -\dfrac{\partial \mathcal{H}}{\partial q}(p(t), q(t), t) \\ \dfrac{\mathrm{d}q}{\mathrm{d}t}(t) = \dfrac{\partial \mathcal{H}}{\partial p}(p(t), q(t), t) \end{cases}$$

En considérant le couple (p,q) comme une unique variable d'espace

$$X = (x_1, \ldots, x_N) = \big((p_1, q_1), \ldots, (p_N, q_N)\big),$$

alors les équations d'Hamilton s'écrivent

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) = \nabla^\perp \mathcal{H}(X(t), t)$$

qu'il faut interpréter comme le système d'équations

$$\forall i \in [\![1,N]\!], \quad \frac{\mathrm{d}x_i}{\mathrm{d}t}(t) = \nabla^\perp_{x_i} \mathcal{H}(X(t), t).$$

Soit Ω un domaine borné simplement connexe. On remarque alors qu'en posant $Z = (z_1, \dots, z_N)$ et

$$\mathcal{H}(Z) = \sum_{i \neq j} a_i a_j G_\Omega(z_i, z_j) + \frac{1}{2} \sum_{i=1}^{N} a_i^2 \widetilde{\gamma}_\Omega(z_i)$$

alors les équations d'Hamilton donnent presque exactement le système point-vortex (1.20). On obtient en vérité :

$$\forall i \in [\![1, N]\!], \quad a_i \frac{\mathrm{d}z_i}{\mathrm{d}t}(t) = \nabla_{x_i}^{\perp} \mathcal{H}(Z(t)).$$

Pour se ramener à l'équation générale, c'est à dire retirer le terme a_i devant $\frac{\mathrm{d}z_i}{\mathrm{d}t}$, on peut par exemple étudier la famille de variables $(z_{i,1}, a_i z_{i,2})_{1 \leq i \leq N}$. Le système point-vortex est donc bien un système Hamiltonien. Dans le plan ou dans les domaines non simplement connexe, c'est aussi le cas. La fonctionnelle $\mathcal{H}$ est appelée Hamiltonien du système. Servons nous de cette structure. Soit $\phi : \Omega^N \times \mathbb{R}_+ \to \Omega$ le flot de ce système différentiel, c'est-à-dire que $t \mapsto \phi(X, t)$ est la solution issue de X. Alors

$$\frac{\mathrm{d}}{\mathrm{d}t} \phi(X, t) = \nabla^{\perp} \mathcal{H}(X(t), t)$$

et donc

$$\nabla \cdot \frac{\mathrm{d}}{\mathrm{d}t} \phi(X, t) = \nabla \cdot \nabla^{\perp} \mathcal{H}(X(t)) = 0.$$

Donc d'après le théorème de Liouville, le flot préserve la mesure. En particulier, pour toute fonction f sur Ω^N, on a donc

$$\int_{\Omega^N} f(X(t)) \mathrm{d}x = \int_{\Omega^N} f(\phi(X, t)) \mathrm{d}X = \int_{\Omega^N} f(X) \mathrm{d}X.$$

Cette propriété nous sera particulièrement utile pour l'étude des collisions.

Quantités conservées

Puisque le système point-vortex est Hamiltonien, il est conservatif : le Hamiltonien est préservé au cours du temps. En effet,

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{H}(Z) = \frac{\mathrm{d}Z}{\mathrm{d}t} \cdot \nabla \mathcal{H}(Z(t), t)) = \nabla^{\perp} \mathcal{H}(Z(t), t)) \cdot \nabla \mathcal{H}(Z(t), t)) = 0.$$

Ainsi, le système point-vortex évolue sur une ligne de niveau du Hamiltonien. Dans le plan (uniquement), le théorème de Noether [66] donne d'autres invariants : le *vecteur de vorticité*

$$M(t) = \sum_{i=1}^{N} a_i z_i(t)$$

et le *moment d'inertie*

$$I(t) = \sum_{i=1}^{N} a_i |z_i(t)|^2.$$

En particulier, lorsque la somme des intensités des point-vortex est non nulle, le *centre de vorticité*

$$B(t) = \left(\sum_{i=1}^{N} a_i \right)^{-1} \sum_{i=1}^{N} a_i z_i(t)$$

est préservé.

Fonction de Robin et dynamique d'un unique point-vortex

Dans cette section nous supposons que $N = 1$. Naturellement, si l'unique point-vortex évolue dans le plan, c'est-à-dire selon le système (1.17), ou sa généralisation (1.27) pour un certain α, alors il est clair que le point-vortex ne se déplace pas, concluant l'étude. Nous nous intéressons donc ici uniquement au cas où le point-vortex est placé dans un domaine borné, que l'on suppose simplement connexe pour simplifier notre discussion. Nous avons déjà mentionné en section 1.2.3 que le point-vortex évolue sur une ligne de niveau du Hamiltonien. Ici, le Hamiltonien n'est autre, à constante près, que la fonction de Robin $\widetilde{\gamma}_\Omega$, dont on a donné la définition à la relation (1.21). La dynamique d'un unique point-vortex est donc

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) = \frac{a}{2} \nabla^\perp \widetilde{\gamma}_\Omega(z(t)). \tag{1.28}$$

Un point-vortex seul évolue sur une ligne de niveau de la fonction de Robin. Celle-ci est étudiée grandement dans [40] ou [11] par exemple. Elle satisfait

$$\widetilde{\gamma}_\Omega(x) \underset{x \to \partial\Omega}{\longrightarrow} +\infty. \tag{1.29}$$

Ainsi, un unique point-vortex se déplaçant dans un domaine borné vérifie l'existence de $\eta > 0$ tel que $d(z(t), \partial\Omega) \geq \eta$. En particulier, la solution est donc toujours globale en temps. Une autre conséquence de cette propriété est qu'il existe toujours x_0 tel que

$$\widetilde{\gamma}_\Omega(x_0) = \min_{x \in \Omega} \widetilde{\gamma}_\Omega(x).$$

Or, un tel point étant nécessairement un point critique de la fonction de Robin, c'est-à-dire

$$\nabla \widetilde{\gamma}_\Omega(x_0) = 0,$$

ceci donne l'existence dans tout domaine borné simplement connexe d'une solution stationnaire au système point-vortex. La relation (1.29) est toujours vraie dans un domaine non simplement connexe (voir [40]). Puisque les autres termes intervenant dans la dynamique (1.23) pour $N = 1$ sont bornés, alors on a toujours pour les mêmes raisons l'existence d'un point critique du Hamiltonien, qui est donc un point stationnaire, dans tout domaine borné.

Notons que la fonction de Robin est sous-harmonique (voir [40]). Ceci implique qu'elle ne possède pas de maximum local.

1.2.4 Désingularisation

Insistons encore une fois, nous n'avons pas *prouvé* qu'une somme de masses de Dirac de tourbillon se déplace dans le fluide selon le système point-vortex. Nous avons *choisi* la dynamique en retirant un terme singulier. Nous allons donc aborder le problème de *désingularisation.* Le principe est de comparer les solutions lisses des équations d'Euler dont la solution initiale est très concentrée autour d'une donnée singulière, à la solution du système point-vortex issue de cette donnée singulière, prouvant ainsi le bien fondé de ce système dynamique. Le premier théorème de désingularisation est le suivant.

On considère un tourbillon initial $\omega_{0,\varepsilon}$ qui est une somme de *poches de tourbillons* s'écrivant

$$\begin{cases} \omega_{0,\varepsilon} = \varepsilon^{-2} \displaystyle\sum_{i=1}^{N} \mathbb{1}_{\Lambda_\varepsilon^i}, \\ |\Lambda_\varepsilon^i| = \varepsilon^2 a_i \end{cases} \tag{1.30}$$

où les Λ_ε^i sont des sous ensembles de $\mathbb{R}^2$ deux à deux disjoints, $\mathbb{1}_{\Lambda_\varepsilon^i}$ désigne la fonction indicatrice de l'ensemble Λ_ε^i et $|\Lambda_\varepsilon^i|$ est sa mesure de Lebesgue. Le théorème de Yudovitch donne l'existence d'ensembles $\Lambda_\varepsilon^i(t)$ tels que

$$\begin{cases} \omega_\varepsilon(t) = \varepsilon^{-2} \sum_{i=1}^N \mathbb{1}_{\Lambda_\varepsilon^i(t)}, \\ |\Lambda_\varepsilon^i(t)| = \varepsilon^2 a_i. \end{cases}$$

On introduit

$$M_\varepsilon^i(t) = \varepsilon^{-2} \int_{\Lambda_\varepsilon^i} x \mathrm{d}x$$

et

$$I_\varepsilon^i(t) = a_i \varepsilon^{-2} \int_{\Lambda_\varepsilon^i} |x - M_\varepsilon^i(t)|^2 \mathrm{d}x.$$

Théorème 1.2.1 (Marchioro, Pulvirenti, 1983, [60])**.** *Supposons que la donnée initiale $\omega_{0,\varepsilon}$ définie par* (1.30) *converge faiblement en mesure lorsque $\varepsilon \to 0$ vers $\sum_{i=1}^N a_i \delta_{z_i}$, c'est-à-dire qu'il existe des points z_i tels que pour toute fonction f continue et bornée,*

$$\lim_{\varepsilon \to 0} \int f(x) \omega_{0,\varepsilon}(x) \mathrm{d}x = \sum_{i=1}^N a_i f(z_i).$$

Supposons de plus que

$$\min_{i \neq j} \operatorname{dist}(\Lambda_\varepsilon^i, \Lambda_\varepsilon^j) \geq \frac{1}{2} \min_{i \neq j} |z_i - z_j| > 0$$

et qu'il existe une constante C indépendante de ε telle que pour tout $i \in [\![1, N]\!]$,

$$I_\varepsilon^i(0) \leq C \varepsilon^2.$$

Alors il existe un temps $T > 0$ tel que pour tout $t \in [0, T]$, pour toute fonction f continue et bornée,

$$\lim_{\varepsilon \to 0} \int f(x) \omega_\varepsilon(x, t) \mathrm{d}x = \sum_{i=1}^N a_i f(z_i(t)),$$

où $t \mapsto z_i(t)$ est la solution du système point-vortex (1.17) *issue des points z_i et d'intensités a_i. De plus,*

$$\lim_{\varepsilon \to 0} M_\varepsilon^i(t) = z_i(t)$$

pour tout $t \in [0, T]$.

Ce théorème prouve donc la convergence faible au sens des mesures de la solution des équations d'Euler vers le système point-vortex lorsque la donnée initiale devient singulière. Il justifie le choix de ce modèle. En revanche, ce théorème n'offre la convergence qu'en temps court, et seulement pour les poches de tourbillons. Turkington [78] prouve que dans un domaine borné et lorsque $N = 1$, c'est-à-dire que l'on considère une donnée initiale se concentrant autour d'un seul point, alors les conclusions de ce théorème sont toujours vraies et cette fois pour T arbitrairement grand. Marchioro [59] étend son propre théorème au temps arbitrairement grand pour N quelconque mais en supposant que les masses sont toutes positives.

Dans l'article [63], les auteurs s'affranchissent de l'hypothèse imposant au tourbillon initial d'être une somme de poches de tourbillons. Le résultat prend alors la forme suivante.

Théorème 1.2.2 (Marchioro, Pulvirenti, 1993, [63])**.** *Soit $(z_i)_{1 \leq i \leq N} \in \mathcal{A}$, (a_i) des intensités. On note $(z_i(t))_{1 \leq i \leq N}$ la solution du système point-vortex* (1.17)*. Soit $T > 0$. On suppose que cette solution est bien définie sur $[0, T]$.*

Pour tout $\varepsilon > 0$, soit $\omega_{\varepsilon,0}$ satisfaisant

$$\begin{cases} \omega_{\varepsilon,0} = \sum_{i=1}^{N} \omega_{\varepsilon,0}^i \quad \textit{et} \quad \operatorname{supp} \omega_{\varepsilon,0}^i \subset D(z_i, \varepsilon) \\ \omega_{\varepsilon,0}^i \textit{ a un signe} \quad \textit{et} \quad \int \omega_{\varepsilon,0}^i \mathrm{d}x = a_i \\ |\omega_{\varepsilon,0}| \leq C\varepsilon^{-\eta}, \quad \textit{où } \eta < 8/3. \end{cases} \tag{1.31}$$

Alors, on notant ω_ε la solution des équations d'Euler telle que $\omega_\varepsilon(\cdot, 0) = \omega_{\varepsilon,0}$, pour tout $d > 0$ il existe $\varepsilon_0(d, T)$ tel que pour tout $\varepsilon < \varepsilon_0$ et pour tout $t \in [0, T]$ on ait

$$\operatorname{supp} \omega_\varepsilon(\cdot, t) \subset \bigcup_{i=1}^{N} D(z_i(t), d),$$

et $d \to 0$ lorsque $\varepsilon \to 0$. De plus, on a pour tout $t \in [0, T]$,

$$\omega_\varepsilon(\cdot, t) \underset{\varepsilon \to 0}{\longrightarrow} \sum_{i=1}^{N} a_i \delta_{z_i(t)}$$

faiblement au sens des mesures.

Une autre façon de donner un théorème de désingularisation, dont le résultat sera cette fois toujours global en temps, est de construire des solutions stationnaires, arbitrairement concentrées. Ceci est simple à construire dans le plan entier pour $N = 1$. En effet, tout tourbillon radial génère un champ de vitesse orthoradial. Ainsi, le tourbillon est transporté par un champ orthoradial. Étant radiale, elle est donc constante en temps. Mais dans un domaine borné ce résultat est non trivial. Rappelons que nous avons établi en section (1.2.3) l'existence de points stationnaires dans tout domaine borné. L'article [74] propose cette construction pour $N = 1$, et pour quelques autres exemples précis.

Théorème 1.2.3 (Smets, Van Schaftingen, 2010, [74])**.** *Soit Ω borné et $a > 0$. Alors il existe $\omega_{0,\varepsilon}$ et x_ε satisfaisant*

$$\operatorname{supp} \omega_{0,\varepsilon} \subset D(x_\varepsilon, C\varepsilon),$$

où C ne dépend pas de ε, telle que la solution des équations d'Euler issue de cette condition initiale soit stationnaire. De plus,

$$\widetilde{\gamma}(x_\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \min \widetilde{\gamma}$$

et

$$\int \omega_{\varepsilon,0}(x) \mathrm{d}x \underset{\varepsilon \to 0}{\longrightarrow} a.$$

L'article [16] propre une généralisation de ce résultat aux solutions stationnaires du système point-vortex avec N points.

Dans le cas des équations SQG et du système point vortex généralisé associé (1.27), il existe aussi des théorèmes de désingularisation. On pourra se référer par exemple à [29], [35] et enfin [73].

Un autre aspect justifiant la pertinence du système point-vortex est la convergence réciproque, celle du système-point vortex vers les équations d'Euler lorsque le nombre de points-vortex tend vers l'infini. L'article [36] montre cette convergence. D'un point de vue numérique, on peut donc simuler les équations d'Euler en remplaçant une tourbillon lisse par une grille de points-vortex. Bien entendu, l'intérêt est encore plus net lorsque le tourbillon est très concentré : l'approximation par un petit nombre fixe de points-vortex est possible. L'article [8] propose également de simuler un éventuel bord du domaine par des points-vortex fixes pour un gain de complexité dans les simulations numériques.

Notons enfin que le système point-vortex intervient dans des problèmes théoriques sur les équations d'Euler, notamment l'existence et la non-unicité des solutions suffisamment faibles. Citons pour référence [37] et [26].

Chapitre 2

État de l'art et contributions

Dans ce chapitre nous présentons les différents problèmes que nous étudierons dans ce mémoire. Nous passons en revue les résultats existants, puis présentons les contributions de la thèse à chacun de ces problèmes en donnant les énoncés des théorèmes principaux ainsi que les idées principales des preuves.

2.1 Confinement autour d'un point stationnaire super-stable

Nous avons vu que le système point-vortex est une limite, dans un certain sens, de solutions du système point-vortex. Nous avons appelé *désingularisation* la justification de cette limite. La question que l'on se pose dans cette section est la suivante : étant donné un tourbillon initial fortement concentré, combien de temps l'approximation de la solution des équations d'Euler par la solution du système point-vortex reste-t-elle valable ? Autrement dit, on essaie de *quantifier* la désingularisation.

2.1.1 Problème de confinement du tourbillon

Nous avons parlé du problème de désingularisation en section 1.2.4. Nous y avons vu, au Théorème 1.2.1 qu'à T et d fixé, il existe $\varepsilon_0(d,T)$ tel que pour $\varepsilon < \varepsilon_0$, un tourbillon initial confiné à ε près autour d'une donnée initiale singulière reste confiné sur $[0,T]$ à d près autour de la solution du système point-vortex.

Dans l'article [17], les auteurs obtiennent un résultat plus fort, formulé un peu différemment. Quel que soit $\varepsilon > 0$ le confinement initial du tourbillon, quel que soit $T > 0$ et quel que soit $\beta \in]0, 1/2[$, alors il existe une constante $C(\alpha,T)$ tel que le confinement a lieu sur l'intervalle de temps $[0,T]$ pour $d = C(\alpha,T)\varepsilon^\beta$. Autrement dit, on ne cherche plus $\varepsilon(d,T)$, mais cette fois on cherche $d(\varepsilon,T)$. Cette formule fait apparaître une fonction puissance de ε.

Naturellement, on peut encore reformuler le problème en cherchant cette fois $T(\varepsilon,d)$ le temps maximal ou le confinement est réalisé. Le résultat précédent nous pousse à poser $d = \varepsilon^\beta$, pour $\beta < 1/2$.

Nous étudions maintenant le problème de confinement formulé comme dans les articles [13] et [18]. On pose $d = \varepsilon^\beta$, avec $\beta < 1/2$, et ω_0 satisfaisant

$$\begin{cases} \omega_0 = \displaystyle\sum_{i=1}^N \omega_0^i \quad \text{et} \quad \operatorname{supp} \omega_0^i \subset D(z_i, \varepsilon) \\ \omega_0^i \text{ a un signe} \quad \text{et} \quad \displaystyle\int \omega_0^i \mathrm{d}x = a_i \\ |\omega_0| \leq C\varepsilon^{-\eta}, \quad \text{où } \eta > 0, \end{cases} \tag{2.1}$$

et l'on suppose que la solution du système point-vortex $t \mapsto (z_i(t))_i$ est globale en temps. On définit le temps de sortie du disque de rayon ε^β par

$$\tau_{\varepsilon,\beta} = \sup \left\{ t \geq 0, \forall s \in [0,t], \operatorname{supp} \omega(\cdot, s) \subset \bigcup_{i=1}^N D(z_i(s), \varepsilon^\beta) \right\}.$$

C'est la quantité que l'on cherche à minorer en fonction de ε.

L'article [13] donne à ce sujet plusieurs résultats. Le premier peut être énoncé ainsi.

Théorème 2.1.1 (Buttà, Marchioro, 2018, [13]). *Si $\beta < 1/2$ et ω_0 satisfait* (2.1)*, alors il existe une constante $\xi_0 > 0$ tel que pour tout ε assez petit,*

$$\tau_{\varepsilon,\beta} > \xi_0 |\ln \varepsilon|.$$

On s'intéresse ensuite à un cas particulier. On se place cette fois dans le disque unité, et l'on suppose que ω_0 satisfait (2.1) pour $N = 1$ et $z_1 = 0$. C'est-à-dire qu'on regarde le problème de confinement au centre d'un disque.

Théorème 2.1.2 (Buttà, Marchioro, 2018, [13]). *Si $\beta < 1/2$ et ω_0 est construite comme évoqué ci-dessus, alors il existe une constante $\xi_0 > 0$ tel que pour tout ε assez petit,*

$$\tau_{\varepsilon,\beta} > \varepsilon^{-\xi_0}.$$

Dans ce cas particulier, la borne pour le temps de confinement est bien meilleure. L'article [13] prouve également que si la donnée initiale du système point-vortex est une solution auto-similaire dont les points s'écartent les uns des autres, alors cette même borne peut être obtenue.

L'objectif de la section 2.1 est d'étendre le résultat de confinement au centre d'un disque à des domaines bornés plus généraux, en choisissant astucieusement le point de confinement. Notons que le Théorème 2.1.1 a été étendu dans l'article [18] au problème de confinement pour les équations SQG, formulé de la même manière.

2.1.2 Point-vortex stationnaire stable et super-stable

Soit Ω un domaine borné simplement connexe. D'après le théorème de représentation conforme de Riemann, un tel domaine satisfait l'existence de biholomorphismes $T : \Omega \to D = D(0,1)$. Ces biholomorphismes sont nombreux, puisqu'ils s'obtiennent tous en composant par un biholomorphisme du disque, famille à trois paramètres réels bien connue. En particulier, quel que soit le point $x_0 \in \Omega$, il existe un biholomorphisme $T : \Omega \to D$ tel que $T(x_0) = 0$. On peut également si besoin imposer $T'(x_0) \in \mathbb{R}_+$. Le biholomorphisme est alors uniquement déterminé. De plus, la fonction de Green du domaine Ω est obtenue par composition de la fonction de Green G_D du disque unité et du biholomorphisme T, comme détaillé en annexe, Proposition A.2.1.

Rappelons qu'en section 1.2.3, nous avons prouvé l'existence pour tout domaine d'un point x_0, point critique du Hamiltonien et donc point stationnaire de la dynamique du système point-vortex pour $N = 1$, écrite à l'équation (1.28). En dérivant l'équation obtenue en Proposition A.2.1, on peut obtenir une relation liant $T''(x_0)$ à $\nabla\widetilde{\gamma}_\Omega(x_0)$. Plus précisément, c'est la Proposition 3.2.4, les trois relations suivantes sont équivalentes :

(*i*) x_0 est un point stationnaire de la dynamique (1.28),

(*ii*) $\nabla\widetilde{\gamma}_\Omega(x_0) = 0$,

(*iii*) $T''(x_0) = 0$.

En liant les deux notions ensemble on peut alors affirmer que pour tout domaine borné simplement connexe, il existe un point x_0 et un biholomorphisme $T : \Omega \to D$ tel que $T(x_0) = T''(x_0) = 0$.

Le calcul peut être poussé plus loin, et l'on peut pour un tel point x_0 lier $T'''(x_0)$ à $D^2\widetilde{\gamma}_\Omega(x_0)$. On a :

$$\frac{1}{2}D^2\widetilde{\gamma}_\Omega(x_0) = \frac{1}{4\pi}\begin{pmatrix} 2\mu^2+p & q \\ q & 2\mu^2-p \end{pmatrix},$$

où

$$\begin{cases} \mu^2 = |T'(x_0)|^2 \\ p = \dfrac{1}{\mu^2}T'''(x_0)\cdot T'(x_0) \\ q = \dfrac{1}{\mu^2}(T'''(x_0))^\perp \cdot T'(x_0). \end{cases}$$

Cette matrice Hessienne a des valeurs propres strictement positives lorsque $|T'''(x_0)| < 2|T'(x_0)|^3$ et de signe strictement opposé lorsque $|T'''(x_0)| > 2|T'(x_0)|^3$. Lorsque les valeurs propres sont strictement positives, x_0 est un minimum local de $\widetilde{\gamma}_\Omega$, et donc le point x_0 est *stable* pour la dynamique. En effet, tout point-vortex placé suffisamment proche de x_0 restera indéfiniment proche de x_0. Si les valeurs propres sont de signe opposés, au contraire, le point x_0 est *instable*, et il existe un voisinage de x_0 et des données initiales arbitrairement proches de x_0 telle que le point-vortex quitte ce voisinage.

Malheureusement, pour obtenir un résultat de confinement autour du point x_0, la stabilité ne nous est pas suffisante. Nous avons besoin d'un résultat de symétrie. Nous appelons cette condition *super-stabilité*. Si $T'''(x_0) = 0$, alors quel que soit $T'(x_0)$ on a

$$\frac{1}{2}D^2\widetilde{\gamma}_\Omega(x_0) = \lambda I_2,$$

matrice dont les valeurs propres sont égales, et donc les lignes de niveau de $\widetilde{\gamma}_\Omega$ proches de x_0, qui sont en première approximation toujours des ellipses, sont en fait des cercles. Cette propriété de symétrie est essentielle dans notre étude pour annuler la singularité dans la loi de Biot-Savart.

Définition 2.1.3. *On dit donc que le point x_0 est un point stationnaire* super-stable *pour la dynamique* (1.28) *s'il existe un biholomorphisme $T : \Omega \to D$ tel que $T(x_0) = T''(x_0) = T'''(x_0) = 0$, ou de manière équivalente, si $\nabla\widetilde{\gamma}_\Omega(x_0) = 0$ et $D^2\widetilde{\gamma}_\Omega(x_0) = \lambda I_2$.*

2.1.3 Confinement en temps long

Au Chapitre 3, nous prouvons le résultat suivant.

Théorème 2.1.4. *Soit $\Omega \in C^{1,1}$ borné simplement connexe et $x_0 \in \Omega$ tels que x_0 soit super-stable pour la dynamique* (1.28)*. Soit ω_0 satisfaisant* (2.1) *pour $N = 1$ et $z_1 = x_0$. Alors pour tout $\beta < 1/2$, pour tout $\xi_0 < \min(\beta, 2-4\beta)$ et pour tout ε assez petit,*

$$\tau_{\varepsilon,\beta} > \varepsilon^{-\xi_0}.$$

La preuve utilise des méthodes inspirées des articles [13] et [51]. On remplace la présence du bord par celle d'un champ extérieur F. La super-stabilité du point x_0 intervient pour estimer la constante de Lipschitz de ce champ extérieur. En effet, si l'on a pour tout x, x' proches de x_0

$$|F(x,t) - F(x',t)| \leq D_t|x - x'|,$$

alors c'est la petitesse de D_t qui donne la forme de la borne de confinement. Si D_t est bornée, alors on obtient une borne de confinement logarithmique en ε. C'est le cas pour le problème de confinement général à N points. Au centre d'un disque, Marchioro et Buttà calculent explicitement que $D_t \leq \delta^2$ si $x, x' \in D(0,\delta)$, et ainsi obtiennent la borne de confinement en puissance de ε. Ces auteurs obtiennent également la même borne dans les cas des configurations auto-similaires, lorsque les points s'éloignent, et qu'alors on a une décroissance explicite de D_t vers 0, lorsque t tend vers l'infini.

C'est la super-stabilité du point x_0 qui nous donne dans notre cas que $D_t \leq \delta$, ce qui est suffisant pour conclure à la même borne puissance. Détaillons l'obtention de cette borne. Le champ extérieur s'écrit

$$F(x,t) = \int \nabla_x^\perp \gamma_\Omega(x,y)\omega(y,t)\mathrm{d}y.$$

Il s'agit donc de contrôler $\left|\nabla_x^\perp \gamma_\Omega(x,y) - \nabla_x^\perp \gamma_\Omega(x',y)\right|$. Supposons pour l'instant seulement que x_0 est un point stationnaire. On utilise la Proposition 4.2.5 ainsi que la décomposition (1.7) pour obtenir, en passant en notations complexes

$$\begin{aligned}
\nabla_x \gamma_\Omega(x,y) &= \nabla_x \left[\gamma_D(T(x),T(y)) + \frac{1}{2\pi}\ln\frac{T(x)-T(y)}{x-y}\right] \\
&= \frac{\overline{T'}(x)(T(x)-T(y))}{2\pi|T(x)-T(y)|^2} - \frac{\overline{T'}(x)(T(x)-T(y)^*)}{2\pi|T(x)-T(y)^*|^2} - \frac{(x-y)}{2\pi|x-y|^2} \\
&= \frac{\overline{T'(x)}}{2\pi\overline{(T(x)-T(y))}} - \frac{\overline{T'(x)}}{2\pi\overline{(T(x)-T(y)^*)}} - \frac{1}{2\pi\overline{(x-y)}}.
\end{aligned} \tag{2.2}$$

Un calcul soigneux donne alors

$$\overline{\nabla_x\gamma(x,y)} - \overline{\nabla_x\gamma(z,y)} = (x-z)\left(\frac{T'''(x_0)}{6\pi T'(x_0)} + \mathcal{O}(\delta)\right),$$

ce qui donne l'estimation du champ extérieur voulue si et seulement si $T'''(x_0) = 0$, expliquant la nécessité de la super-stabilité.

Les quantités importantes qui interviennent sont le centre de vorticité

$$B(t) = \int \omega(x,t)\mathrm{d}x,$$

le centre d'inertie

$$I(t) = \int |x-B(t)|^2\omega(x,t)\mathrm{d}x$$

et enfin les moments plus généraux, d'ordre $4n$:

$$m_n(t) = \int |x-B(t)|^{4n}\omega(x,t)\mathrm{d}x.$$

Le contrôle de D_t donne pour tout $t < \min(\tau_{\varepsilon,\beta}, \varepsilon^{-\beta})$:

$$|B(t) - x_0| \leq C\varepsilon$$

et

$$|I(t)| \leq C\varepsilon^2$$

ainsi qu'un contrôle des moments d'ordre $4n$. On estime ensuite la croissance du rayon R_t du support du tourbillon $\omega(\cdot,t)$ en établissant la majoration suivante. Soit $t \leq \tau_{\varepsilon,\beta}$. Soit $s \mapsto X_t(s)$ une trajectoire, au sens de flot de l'équation d'Euler, telle que $|X_t(t)| = R_t$, c'est-à-dire que la particule de fluide que l'on considère se trouve au temps t au bord du support du tourbillon. On a alors

$$\frac{\mathrm{d}}{\mathrm{d}s}|X_t(s) - B(s)|\Big|_{s=t} \leq C\varepsilon^\beta R_t + \frac{C}{R_t^3}I(t) + C\left(\varepsilon^{-\eta}\int_{|x-B|>R_t/2}\omega(x,t)\mathrm{d}x\right)^{1/2}.$$

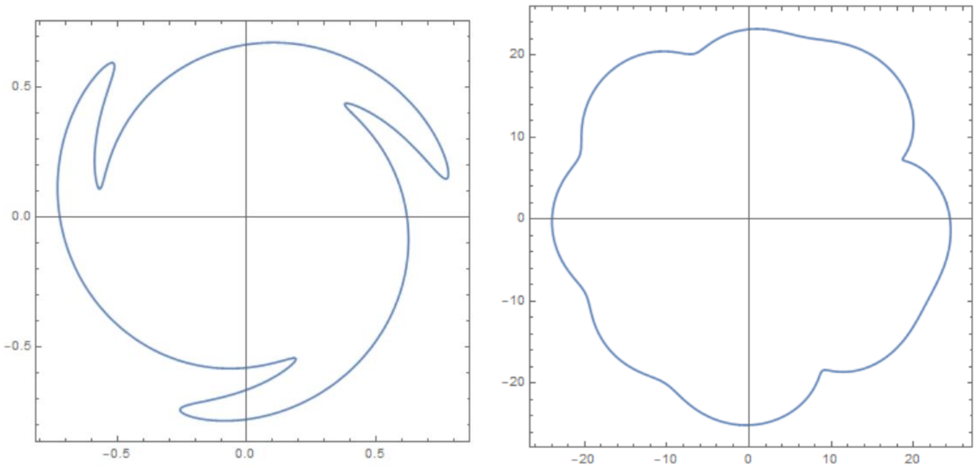


FIGURE 2.1 – Deux exemples de domaines dont 0 est un point super-stable.

En reliant la quantité $\int_{|x-B|>R_t/2}\omega(x,t)\mathrm{d}x$ aux moments d'ordre $4n$, et utilisant les estimations des moments obtenues précédemment, on en déduit que R_t est contrôlé par la solution f de l'équation différentielle :

$$\begin{cases} f'(t) = C\varepsilon^\beta f(t) + \frac{C\varepsilon^2}{f^3(t)} \\ f(0) = 4\varepsilon. \end{cases}$$

La résolution de cette équation et un raisonnement par l'absurde sur la croissance du support permettent de conclure.

Des exemples de domaines possédant un point-super stable sont donnés au Chapitre 3, citons-les :
— tout domaine simplement connexe invariant par rotation d'un angle $\theta \not\equiv 0\,[\pi]$;
— tout domaine Ω image d'un biholomorphisme $f : D \to \Omega$ s'écrivant de la façon suivante :

$$f(x) = a_1 x + \sum_{k=4}^{\infty} a_k x^k.$$

Pour les domaines précédemment cités, 0 est un point super stable. Avec cette dernière formule, on peut en particulier construire des domaines sans symétries ni invariance par rotation. Ces deux méthodes nous permettent de construire deux exemples illustrés en Figure 2.1.

2.2 Collisions de points vortex

Dans la section 1.2.1, nous avons vu qu'il est possible en théorie qu'une solution maximale du système point-vortex ne soit pas globale en temps, et que ceci correspond à une collision à un temps T^* fini, soit entre deux points-vortex, soit d'un point-vortex au bord. Ces collisions sont définies *a priori* par les relations

$$\exists i \neq j, \quad \liminf_{t\to T^*} |z_i(t) - z_j(t)| = 0$$

ou

$$\exists 1 \leq i \leq N, \quad \liminf_{t\to T^*} d(z_i(t), \partial\Omega) = 0.$$

Nous verrons par la suite qu'avec de bonnes hypothèses sur les intensités des points-vortex, on peut remplacer ces $\liminf$ par de vraies limites.

2.2.1 Existence de collisions

L'existence de collisions de point-vortex a été prouvée indépendamment par [39], [67], et [2]. Il est possible de construire une collision explicite à trois points vortex. L'article [39], le premier que l'on connaisse aujourd'hui établissant l'existence d'une collision, est discuté et traduit dans [6]. Donnons un exemple d'une telle collision, que l'on peut trouver par exemple dans [62]. En prenant des intensités $a_1 = 2$, $a_2 = 2$ et $a_3 = -1$, et des positions initiales $z_1 = (-1, 0)$, $z_2 = (1, 0)$ et $z_3 = (1, \sqrt{2})$, on obtient que les distances $l_{i,j} = |z_i(t) - z_j(t)|$ entre les points-vortex satisfont

$$l_{i,j}^2(t) = l_{i,j}^2(0)\sqrt{1 - \frac{t}{3\sqrt{2}\pi}}.$$

On voit donc en particulier dans cette formule que $l_{i,j}(T^*) = 0$ au temps $T^* = 3\sqrt{2}\pi$.

L'article [68] donne des collisions à quatre et cinq vortex. Les articles [2] et [4] étudient plus en détail les collisions à trois vortex, et enfin [54] donne une condition nécessaire et suffisante sur la donnée initiale pour obtenir une collision lorsque $N = 3$. En particulier, ces collisions sont nécessairement *auto-similaires* (voir [54] et [45]). On dit qu'une dynamique est auto-similaire s'il existe un point B, et des fonctions l et θ telles que

$$z_i(t) - B = (z(0) - B)l(t)e^{i\theta(t)}.$$

Les rapports des longueurs et des angles entre les points sont donc préservés. Pour $N = 3$ par exemple, ceci implique que le triangle $(z_1(t), z_2(t), z_3(t))$ est semblable au triangle $z_1(0), z_2(0), z_3(0)$ pour tout temps.

L'article [10] donne l'existence de collisions à trois vortex pour le système point-vortex issu des équations SQG avec $s = 1/2$, c'est-à-dire l'α-modèle pour $\alpha = 2$. De plus, contrairement au cas des équations d'Euler, il est prouvé que des collisions non auto-similaires peuvent se produire. L'article [72] prouve explicitement l'existence de collisions, cette fois auto-similaires, pour ce même système. En suivant la même méthode, nous prouverons au Chapitre 5 correspondant à l'article [21] l'existence de collisions pour les α-modèles, et ce pour tout $\alpha > 0$. Ces collisions obtenues sont auto-similaires.

Citons pour conclure l'article [37]. Il y est prouvée l'existence de *burst*, que l'on pourrait traduire par *éclatement*, de point-vortex. L'éclatement d'un point-vortex est en quelque sorte l'inverse temporel d'une collision. À l'instant $t = 0$, on considère un unique point-vortex z, et l'on construit une solution du système point-vortex pour les temps $t > 0$ à plusieurs points-vortex z_i *issus* de z, c'est-à-dire satisfaisant

$$\lim_{t \to 0} z_i(t) = z.$$

Le théorème obtenu dans [37] affirme que dans le plan ainsi que dans tout domaine borné, de n'importe quelle configuration initiale est issue un éclatement de vortex. C'est-à-dire que l'on peut construire une solution dont l'un des vortex éclate en trois vortex au sens que l'on vient de définir. Ceci donne un résultat d'existence théorique de collisions, et prouve l'existence de collision dans tout domaine borné.

2.2.2 Improbabilité des collisions

Dans la suite, nous désignons les point-vortex par la notation $x_i(t)$ plutôt que $z_i(t)$ pour coïncider avec la façon usuelle de les nommer dans ce genre d'étude.

Nous venons de voir que des collisions sont possibles, et donc en particulier les solutions du système point-vortex ne sont pas toujours globales en temps. Cependant, on peut se demander avec quelle probabilité des points-vortex placés au hasard collisionnent-t-ils. La réponse dans tous les cas connus est 0. C'est-à-dire que les collisions sont *improbables*. Précisons ces résultats.

Dans le tore, les intensités étant fixées, la mesure de Lebesgue de l'ensemble des données initiales menant à une collision est nulle. Ce théorème a été prouvé dans [23]. Dans le disque unité, ce résultat est toujours vrai [61]. Dans le cas du plan entier, le résultat d'improbabilité a été prouvé dans [61] mais seulement lorsque les intensités satisfont une condition de *non neutralité des clusters*

$$\forall P \subset \{1, \dots, N\}, \quad P \neq \emptyset, \quad \sum_{i \in P}^{N} a_i \neq 0. \tag{2.3}$$

Cette hypothèse a été affaiblie dans [34] à la condition de *non neutralités des sous clusters*

$$\forall P \subset \{1, \dots, N\}, \quad P \neq \emptyset, \quad P \neq \{1, \dots, N\}, \quad \sum_{i \in P}^{N} a_i \neq 0. \tag{2.4}$$

Le résultat est également étendu par [29] et [34] aux α-modèles.

L'improbabilité des collisions est également vraie, sans hypothèse sur les intensités, pour les domaines bornés généraux. Nous avons le théorème suivant, prouvé au Chapitre 4.

Théorème 2.2.1. *Soit Ω un domaine borné tel que $\partial\Omega \in C^{2,\alpha}$ pour un certain $\alpha > 0$. Soit $(a_i)_{1\leq i\leq N} \in \mathbb{R}^N$ des intensités et $(\xi_j)_{1\leq j\leq m}$ des circulations. Alors la mesure de Lebesgue de l'ensemble des positions initiales conduisant le système* (1.20) *à une collision est nulle.*

Ce théorème étend le résultat obtenu dans [61] pour le disque unité. Donnons les idées principales de la preuve du Théorème 2.2.1. La structure de la preuve est similaire à celle présentée dans [61] à deux différences près : il nous faut régulariser la dynamique de façon astucieuse, et il faut obtenir une estimation des fonctions de Green dans les domaines bornés généraux non triviale.

Mesure des ε-collisions

Si $X \in \Omega^N$ est une donnée initiale, on note $t \mapsto S_t X$ la solution du système point-vortex (1.23) issue de X. Soit

$$d(X) = \min\left(\min_{i\neq j} |x_i - x_j|, \min_i \mathrm{d}(x_i, \partial\Omega)\right) \quad \forall X = (x_1, \dots, x_N).$$

On rappelle la définition (1.22) de $\mathcal{A}_\Omega$. On peut alors écrire que

$$\mathcal{A} = \{X \in \Omega^N, \ d(X) > 0\}.$$

On note $\tau(X) \in [0, +\infty]$ le temps de vie d'une solution issue d'une condition admissible, que l'on peut définir comme

$$\tau(X) = \sup\{t \geq 0, \ d(S_t X) > 0\},$$

On définit une *ε-collision* par un temps $\tau_\varepsilon(X)$, dont la définition, donnée précisément en Section 4.4.2 est proche de

$$\tau_\varepsilon(X) \approx \sup\{t \geq 0, \ d(S_t X) > \varepsilon\}.$$

En particulier, et de façon rigoureuse, l'ensemble des données initiales conduisant à une collision, c'est-à-dire $\{X \in \mathcal{A}, \tau(X) < +\infty\}$, satisfait

$$\big\{X \in \mathcal{A}, \tau(X) < +\infty\big\} = \bigcap_{\varepsilon > 0} \big\{X \in \mathcal{A}, \tau_\varepsilon(X) < +\infty\big\}$$

où l'intersection sur les $\varepsilon > 0$ peut naturellement être réduite à une réunion dénombrable d'ensemble. Ainsi, en prouvant que la mesure de l'ensemble $\{X \in \mathcal{A}, \tau_\varepsilon(X) < +\infty\}$ tend vers 0 lorsque ε tend vers 0, nous prouvons l'improbabilité des collisions dans Ω.

Soit S^ε une dynamique régularisée, dont nous détaillerons la construction ci dessous, satisfaisant

$$\begin{cases} \forall X \in \Omega^N, \quad t \mapsto S_t^\varepsilon X \text{ est définie sur } \mathbb{R}_+, \\ \forall t \leq \tau_\varepsilon(X), \quad S_t^\varepsilon X = S_t X. \end{cases}$$

On introduit une fonction ϕ_ε que l'on pourrait qualifier *d'énergie*, satisfaisant une relation proche de

$$d(S_t^\varepsilon X) = \varepsilon \iff \phi_\varepsilon(X) \geq \frac{1}{\varepsilon^\eta}$$

pour un certain $\eta \in]0,1[$, on obtient par une inégalité de Markov que pour tout $\tau \geq 0$,

$$\lambda(\{X \in \Gamma, \tau_\varepsilon(X) \leq \tau\}) \leq C\varepsilon^\eta \int_\Gamma \sup_{t\in[0,\tau]} \phi_\varepsilon(S_t^\varepsilon X) \mathrm{d}\lambda(X).$$

On utilise ensuite le théorème de Liouville pour la dynamique régularisée pour prouver que la quantité

$$\int_\Gamma \sup_{t\in[0,\tau]} \phi_\varepsilon(S_t^\varepsilon X) \mathrm{d}\lambda(X)$$

peut être majorée par $1+\tau$ multiplié par une constante dépendant seulement du domaine Ω, à condition d'avoir les estimations données ci dessous (2.5) et (2.6). Ceci donne le résultat en faisant tendre ε vers 0 puis en prenant une intersection dénombrable de $\tau \to +\infty$.

Estimations d'énergie

Comme suggéré par Marchioro et Pulvirenti dans leur livre [61], la première difficulté est d'obtenir les estimations suivantes pour la fonction de Green, données au Lemme 4.3.7.

$$\iint_{\Omega\times\Omega} \frac{1}{|x-y|^\kappa} |\nabla_x G_\Omega(x,y) \cdot \nabla^\perp \widetilde{\gamma}_\Omega(x)| \mathrm{d}x\mathrm{d}y < +\infty \tag{2.5}$$

et

$$\iint_{\Omega\times\Omega} \frac{1}{d(x,\partial\Omega)^\kappa} |\nabla_x G_\Omega(x,y) \cdot \nabla^\perp \widetilde{\gamma}_\Omega(x)| \mathrm{d}x\mathrm{d}y < +\infty. \tag{2.6}$$

Nous commençons par établir ces estimations pour les domaines simplement connexes. Si $\Omega \subsetneq \mathbb{R}^2$ est simplement connexe, on sait d'après le théorème de représentation conforme de Riemann qu'il existe un biholomorphisme $T : \Omega \to D$, où $D = D(0,1)$. Pour un tel biholomorphisme, on a la relation suivante, prouvée en Annexe A.2.1,

$$G_\Omega(x,y) = G_D(T(x), T(y)).$$

En réalisant les changements de variables $x' = T(x)$ et $y' = T(y)$, et avec l'hypothèse de régularité du bord $\partial\Omega \in C^{2,\alpha}$, on réduit les inégalités (2.5) et (2.6) aux cas où $\Omega = D$; ce cas étant déjà résolu dans [61].

De la même manière, nous prouvons ces estimations pour les domaines extérieurs, c'est-à-dire les domaines dont le complémentaire est simplement connexe. En effet, pour ces domaines, il est toujours possible de se ramener au cas du complémentaire du disque unité, dont la fonction de Green est explicite.

Pour étendre ce résultat aux domaines non simplement connexes, on utilise la décomposition suivante. Notons Γ_j les composantes connexes du bord du domaine, avec $0 \leq j \leq m$, et par convention, Γ_0 est la frontière extérieur. Soit Ω_j le domaine ayant comme bord Γ_j dans lequel est inclus Ω, c'est-à-dire Ω_0 est le domaine intérieur à Γ_0, et Ω_j, pour $j \geq 1$, est le domaine extérieur à Γ_j. Cette décomposition est illustrée en Figure 2.2.

Soit V_j un voisinage Γ_j, comme illustré en Figure 2.3.

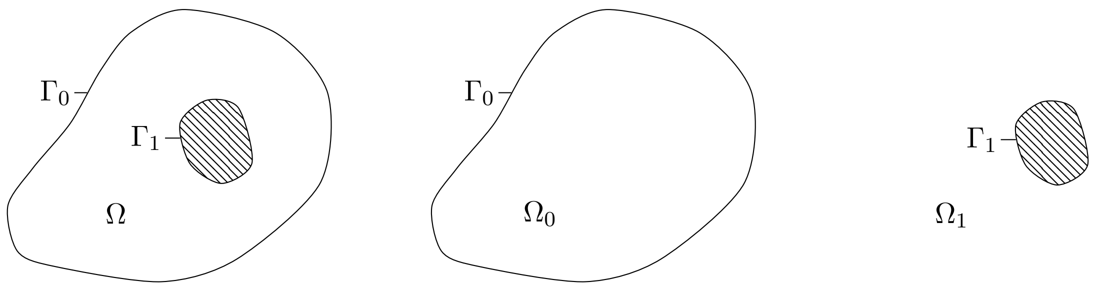


Figure 2.2 – Un exemple de domaines Ω, Ω_0 et Ω_1.

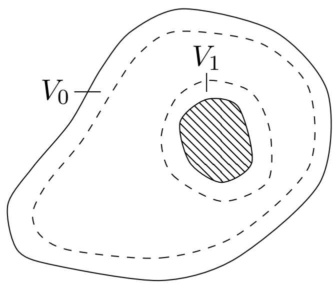


Figure 2.3 – Les domaines (V_j) pour le même domaine Ω qu'en Figure 2.2.

Alors la fonction $\gamma_\Omega - \gamma_{\Omega_j}$ est bornée sur $V_j \times \Omega$. C'est le Lemme 4.3.3 que nous montrerons au Chapitre 4. Ceci permet de prouver les estimations (2.5) et (2.6) pour tout domaine Ω, en se ramenant, au voisinage de chaque bord, au cas où le domaine est simplement connexe, ou un domaine extérieur à un domaine simplement connexe.

Régularisation de la dynamique

La construction de la dynamique régularisée dans [61] nécessite seulement d'introduire une fonction $\ln_\varepsilon : \mathbb{R}_+ \to \mathbb{R}$ régularisée proche de 0. Dans notre cas, de nouvelles difficultés apparaissent, puisque les fonctions de Green et de Robin ne sont plus explicites. On cherche donc à construire des fonctions G_ε et $\widetilde{\gamma}_\varepsilon$ satisfaisant aux conditions suivantes.

(i) La dynamique est globale en temps : il faut donc s'assurer que $G_\varepsilon \in C^2(\overline{\Omega} \times \overline{\Omega})$ et $\widetilde{\gamma}_\varepsilon \in C^1(\overline{\Omega})$ ainsi que $\nabla_x^\perp G_\varepsilon$ et $\nabla^\perp \widetilde{\gamma}_\varepsilon$ soient tangents à $\partial\Omega$.

(ii) La dynamique régularisée est égale à la vraie dynamique sauf lors d'une ε-collision :

$$d((x,y)) \gtrsim \varepsilon \quad \Longrightarrow \quad \begin{cases} G_\varepsilon(x,y) = G(x,y) \\ \widetilde{\gamma}_\varepsilon(x) = \widetilde{\gamma}(x). \end{cases}$$

(iii) Pour que les estimations d'énergie tiennent, il faut également qu'il existe une constante C indépendante de ε telle que

$$\begin{cases} |\widetilde{\gamma}_\varepsilon(x)| \le |\widetilde{\gamma}(x)| \\ |\nabla \widetilde{\gamma}_\varepsilon(x)| \le |\nabla \widetilde{\gamma}(x)| \\ |G_\varepsilon(x,y)| \le |G(x,y)| \\ |\nabla_x G_\varepsilon(x,y)| \le \dfrac{C}{|x-y|}. \end{cases}$$

La façon dont nous obtenons toutes ces conditions est la suivante. On prend une fonction de seuil $f_\varepsilon \in C^\infty(\mathbb{R},\mathbb{R})$ impaire telle que

$$\begin{cases} f_\varepsilon(r) = r, & \forall |r| < \frac{1}{2\pi}|\ln\varepsilon| \\ f_\varepsilon(r) = L_\varepsilon, & \forall r > \frac{1}{2\pi}|\ln\varepsilon| + 1 \\ 0 \leq f'_\varepsilon(r) \leq 1, & \forall r \in \mathbb{R} \end{cases}$$

pour une certaine constante L_ε. On pose alors

$$\widetilde{\gamma}_\varepsilon(x) = f_\varepsilon(\widetilde{\gamma}(x))$$

et

$$\begin{cases} G_\varepsilon(x,y) = f_\varepsilon(G_{\mathbb{R}^2}(x,y)) + f_\varepsilon(\gamma(x,y)) & \text{si } (x,y) \in \Omega\times\Omega,\ \ x \neq y \\ G_\varepsilon(x,y) = 0 & \text{si } x \in \partial\Omega \text{ ou } y \in \partial\Omega \\ G_\varepsilon(x,x) = -L_\varepsilon + f_\varepsilon(\widetilde{\gamma}(x)) & \text{si } x \in \Omega. \end{cases}$$

Si la définition de $\widetilde{\gamma}_\varepsilon$ est assez naturelle, la construction de G_ε l'est un peu moins. Il n'est pas possible de choisir simplement $G_\varepsilon(x,y) = f_\varepsilon(G_\Omega(x,y))$ car les deux singularités des termes $G_{\mathbb{R}^2}(x,y)$ et $\gamma(x,y)$ peuvent se compenser, lorsque x et y convergent vers le même point du bord. Il faut donc appliquer le seuil sur chacun des termes séparément. Nous vérifierons au Chapitre 4 que cette construction convient.

2.3 Convergence et régularité au temps de collision

Bien que nous ayons montré que les collisions de point-vortex sont improbables, des collisions existent néanmoins. Le but de cette section est d'étudier le comportement des point-vortex au temps de la collision. On se demande en particulier si les trajectoires sont bornées, si les points convergent, et la régularité de leur trajectoire au temps de collision.

2.3.1 Théorèmes de borne uniforme

Un premier théorème de régularité est le théorème de borne uniforme suivant. Il prouve que dans le plan, sous hypothèse de non neutralité des clusters, les trajectoires sont bornées sur $[0,T^*[$ même lorsqu'une collision a lieu au temps T^*. En d'autres termes, les collisions à l'infini sont exclues.

Théorème 2.3.1 (Godard-Cadillac, 2022, [34])**.** *Soit $T^* > 0$ et $\alpha > 0$. Soit $t \mapsto X(t)$ une solution de l'α-modèle* (1.27) *satisfaisant l'hypothèse de non neutralité des clusters* (2.3)*. Alors il existe une constante C telle que pour tout X ne produisant pas de collision avant le temps T^*,*

$$\sup_{t\in[0,T^*[} |X(t) - X(0)| \leq C.$$

De plus, pour $\alpha = 1$ c'est-à-dire le système point-vortex (1.17), on a un résultat de convergence donné par le théorème suivant.

Théorème 2.3.2 (Godard-Cadillac, 2022, [34])**.** *Soit $t \mapsto (x_i(t))_{1\leq i\leq N}$ une solution du système point-vortex* (1.17) *définie sur $[0,T^*[$. Alors pour tout $i \in [\![1,N]\!]$, il existe $x_i^* \in \mathbb{R}^2$ tel que*

$$x_i(t) \underset{t\to T^*}{\longrightarrow} x_i^*.$$

Ce théorème exclut des comportements de points-vortex pathologiques à l'instant de la collision. Nous allons étudier au Chapitre 5 ce même problème de convergence dans le cas général des α-modèles, et dans le cas $\alpha = 1$ dans un domaine borné, c'est-à-dire pour les systèmes point-vortex (1.27) et (1.23). Nous allons également obtenir une régularité Hölderienne des trajectoires au temps de collision, sous certaines conditions. Précisons les résultats et des éléments de preuve.

2.3.2 Régularité lors d'une collision dans le plan

On se place dans le plan entier, et l'on étudie l'α modèle (1.27). On a le théorème suivant, prouvé au Chapitre 5.

Théorème 2.3.3. *Soit* $T^* \in]0,1]$ *et* $\alpha \geq 0$. *Soit* $t \mapsto (x_i(t))_{1\leq i\leq N}$ *une solution de l'α-modèle* (1.27) *définie au moins sur* $[0,T^*)$, *une collision étant autorisée au temps* T^*.

(i) Si les intensités vérifient l'hypothèse de non neutralité des clusters (2.3), *alors les trajectoires* $t \mapsto x_i(t)$ *admettent une limite en* T^* *et sont* $\frac{1}{1+\alpha}$*-Hölderiennes sur* $[0,T^*]$ *: pour tout* $i \in \{1,\ldots,N\}$,

$$\forall\, t_1 < t_2 \in [0,T^*), \qquad \big|x_i(t_2) \,-\, x_i(t_1)\big| \;\leq\; C\,|t_2 - t_1|^{\frac{1}{\alpha+1}}.$$

De plus, la constante de Hölder ne dépend pas des conditions initiales.

(ii) Si les intensités vérifient seulement l'hypothèse de non neutralité des sous *clusters* (2.4), *alors pour tout* $i \neq j \in [\![1,N]\!]$ *tels que* $\liminf_{t\to T^*} |x_i(t) - x_j(t)| = 0$, *on a*

$$\forall\, t \in [0,T^*), \qquad \big|x_i(t) - x_j(t)\big| \;\leq\; C\,|T^* - t|^{\frac{1}{\alpha+1}}, \tag{2.7}$$

où la constante C *ne dépend pas des données initiales.*

(iii) Si les intensités satisfont (2.4) *et que tous les points ne collisionnent pas ensemble, c'est-à-dire*

$$\max_{i\neq j}\ \limsup_{t\to T^*}\ |x_i(t) - x_j(t)| > 0,$$

alors pour tout $i \in [\![1,N]\!]$, *les trajectoires* $t \mapsto x_i(t)$ *admettent une limite* x_i^* *en* T^* *et satisfont*

$$\forall\, t \in [0,T^*), \qquad \big|x_i(t) \,-\, x_i^*\big| \;\leq\; C'\,|T^* - t|^{\frac{1}{\alpha+1}},$$

où la constante C' *dépend cette fois, entre autres, de la donnée initiale.*

Dans les trois parties de ce théorème, il y a un résultat commun, la relation (2.7) lorsque les indices $i \neq j$ correspondent à des points collisionnant ensemble. Pour obtenir ce résultat, on établit la proposition suivante.

Proposition 2.3.4 (Condition suffisante pour l'absence de collision)**.** *Soit* $i \in [\![1,N]\!]$, *et* $t \mapsto x_i(t)$ *une famille de* N *trajectoires de points de* $\mathbb{R}^p$ *définie sur* $[0,T^*)$, $T^* > 0$. *On suppose de plus que*

$$\frac{\mathrm{d}}{\mathrm{d}t}x_i \in L^1_{\mathrm{loc}}([0,T^*)).$$

Soit $(a_i)_{1\leq i\leq N} \in \mathbb{R}^N$ *des intensités satisfaisant* (2.4), *de sorte que pour toute partie* $P \subsetneq [\![1,N]\!]$ *l'on puisse définir*

$$B_P(t) = \frac{\sum_{i\in P} a_i x_i(t)}{\sum_{i\in P} a_i}.$$

On suppose que les trajectoires sont telles qu'il existe des constantes $C_0, C_1 \geq 0$ *et* $\alpha \geq 0$ *telles que*

$$\forall P \in \mathcal{P}_0(N) \setminus \{1\ldots N\}, \qquad \left|\frac{\mathrm{d}}{\mathrm{d}t}B_P(t)\right| \leq \sum_{i\in P}\sum_{j\notin P} \frac{C_0}{|x_i(t) - x_j(t)|^\alpha} + C_1. \tag{2.8}$$

Alors il existe une constante $C_2 > 0$ *telle que pour tout* $\eta \in (0,1]$, *pour tout* $t \in [0,T^*)$ *tel que*

$$T^* - t \leq C_2\,\eta^{\alpha+1},$$

et pour tout indices $i,j \in \{1,\ldots,N\}$ *l'implication suivante est vraie :*

$$|x_i(t) - x_j(t)| \geq \eta \quad\Longrightarrow\quad \forall \tau \in [t,T^*), \quad |x_i(\tau) - x_j(\tau)| \geq \frac{\eta}{2}.$$

La constante C_2 *dépendant seulement de* α, *des intensités, de* C_0, C_1 *et* N.

Ce que ce lemme prouve, c'est qu'à distance $\eta > 0$ fixée, à partir d'un certain temps, deux points situés à distance plus grande que η ne peuvent plus collisionner dans le temps imparti. La contraposée de ce résultat est une condition nécessaire de collision : si x_i et x_j collisionnent ensemble au temps T^*, il faut nécessairement que $|x_i(t) - x_j(t)| \leq \eta$ dès que $T^* - t \leq C_2\,\eta^{\alpha+1}$, d'où (2.7).

Nous montrons en section A.3 que la dynamique du système point-vortex satisfait l'hypothèse (2.8).

L'argument central de la preuve de la Proposition 2.3.4 est une *clusterisation*, c'est-à-dire une partition des points en groupes de points proches les uns des autres. On utilise pour cela le Lemme suivant, inspiré de [12], et illustré en Figure 2.4.

Lemme 2.3.5 (Lemme des boules)**.** *Soit* (x_i) *une famille de* N *points de* $\mathbb{R}^p$*,* $p \in \mathbb{N}^*$*. Alors pour tout* $\kappa \in (0,1)$ *et tout* $d > 0$*, il existe* $\delta \in [\left(\frac{\kappa}{8}\right)^N d, d)$ *et une partition* $\mathfrak{P}$ *de l'ensemble* $\{1, \ldots, N\}$ *telle que*

$$\forall\, P \in \mathfrak{P}, \quad \forall\, i,j \in P, \qquad |x_i - x_j| \leq \delta$$

et

$$\forall\, P \neq P' \in \mathfrak{P}, \quad \forall\, i \in P, \quad \forall\, j \in P', \qquad |x_i - x_j| \geq \kappa^{-1}\delta.$$

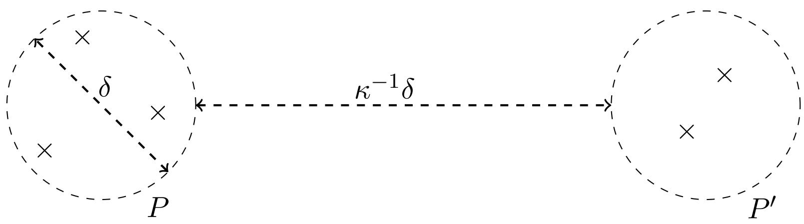


FIGURE 2.4 – Les points d'un cluster sont proches les uns des autres, loin des autres clusters.

Nous raisonnons alors ainsi. Soit t_1 de la forme $T^* - t_1 \leq C_2\eta^{\alpha+1}$, où l'on choisira la constante C_2 à la fin. On suppose que i et j sont tels que $|x_i(t_1) - x_j(t_1)| \geq \eta$. On clusterise avec le Lemme des boule en choisissant κ et d assez petits de sorte que i et j ne soient pas dans le même cluster. On note la partition ainsi obtenue $\mathfrak{P}^1$ et la taille des clusters associée δ_1. Deux cas de figure peuvent se produire. Si tous les points se déplacent peu jusqu'au temps T^*, auquel cas les points x_i et x_j restent loin l'un de l'autre et le résultat est prouvé. Si ce n'est pas le cas, c'est qu'il existe un temps t_2 et un indice k tel que x_k s'est "beaucoup" déplacé entre t_1 et t_2 (Figure 2.5a). Précisément, on définit t_2 comme le premier temps tel qu'il existe k tel que

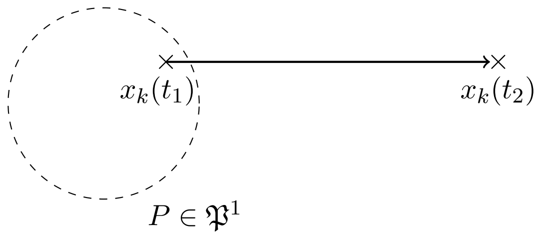
$$|x_k(t_2) - x_k(t_1)| = \kappa^{-1}\delta_1/8. \tag{2.9}$$

On note $P \in \mathfrak{P}^1$ le cluster contenant k. Ce qui se produit alors, c'est qu'à cause de la quasi-conservation du centre de vorticité, exprimée à l'équation (2.8), la vitesse du centre de vorticité est contrôlée par l'inverse de la distance séparant les cluster qui est de l'ordre $\kappa^{-1}\delta$, ce qui est une "grande" quantité. Par conséquent, si le $t_2 - t_1$ est trop court, c'est-à-dire en particulier si C_2 est assez petite, il n'est pas possible que B_P parcourt une distance assez grande pour rester proche de x_k (Figure 2.5b). C'est donc qu'il existe un autre indice $\ell \in P$ dans le même cluster qui s'est également éloigné de x_k (Figure 2.5c).

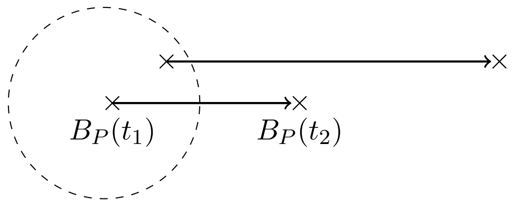
On applique alors une nouvelle fois le Lemme des boules sur les points au temps t_2, obtenant une partition $\mathfrak{P}^2$. Pour ce faire, on choisit astucieusement une distance d suffisamment petite. En particulier, on peut s'assurer que les points d'indices k et ℓ, dont on sait minorer la distance, ne soient plus dans le même cluster dans $\mathfrak{P}^2$ (Figure 2.5d)

Le choix de prendre t_2 le premier temps tel que (2.9) soit vérifié assure que deux points qui appartenaient à deux clusters différents dans $\mathfrak{P}^1$ appartiennent toujours à deux clusters différents dans $\mathfrak{P}^2$. Ainsi, $\mathfrak{P}^2$ est une sous partition de $\mathfrak{P}^1$, qui est stricte car k et ℓ ne sont plus dans le même cluster.

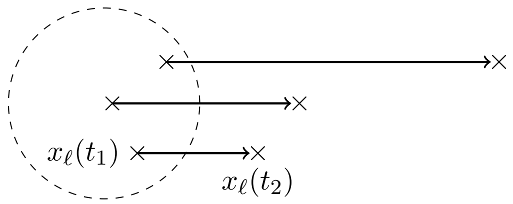
En répétant alors l'opération, construisant ainsi une suite $\mathfrak{P}^q$ de partition des points au temps t_q, le nombre de clusters dans $\mathfrak{P}^q$ est strictement croissant. Ce nombre de clusters ne pouvant dépasser N, le premier cas finit nécessairement par se produire. Puisque i et j ne sont pas dans le même cluster dans $\mathfrak{P}^1$, cela signifie que les points ne collisionnent pas. Un calcul montre alors qu'on peut choisir C_2 indépendant de η et convenant au problème, ce qui termine la preuve.



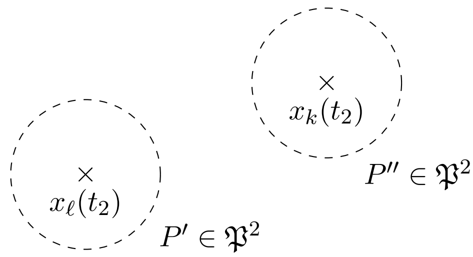
(a) Le point d'indice k s'est beaucoup déplacé.



(b) Le centre de vorticité B_P ne peut pas acquérir une assez grande vitesse pour rester proche de x_k.



(c) C'est donc qu'il existe un point x_ℓ qui s'éloigne lui aussi de x_k.



(d) Ainsi dans la clusterisation au temps t_2, k et ℓ ne sont pas dans le même cluster.

Figure 2.5 – Pour qu'un point puisse parcourir une grande distance, il faut que son cluster se *brise*.

2.3.3 Régularité lors d'une collision dans un domaine borné

Intéressons nous maintenant au cas $\alpha = 1$ et Ω borné. Nous prouvons, toujours au Chapitre 5, le théorème suivant.

Théorème 2.3.6. *Soit* $T^* \in]0,1]$. *Soit* $t \mapsto (x_i(t))_{1 \le i \le N}$ *une solution du système point-vortex* (1.23) *définie sur* $[0, T^*[$ *dans un domaine borné* Ω *à la frontière régulière. Supposons que les intensités* $a_i \neq 0$ *satisfont l'hypothèse de non neutralité des clusters* (2.3). *On pose*

$$I := \big\{ i \in [\![1, N]\!], \ \liminf_{t \to T^*} \ \mathrm{dist}\,\big(x_i(t); \partial\Omega\big) = 0 \big\}.$$

Alors il existe une constante C *telle que*

(i) Si $i \notin I$ *alors*

$$\forall\, t_1 < t_2 \in [0, T^*), \qquad \big|x_i(t_2) - x_i(t_1)\big| \ \le\ C\sqrt{t_2 - t_1}\,.$$

En particulier, la trajectoire converge vers un point intérieur à Ω.

(ii) Si $i \in I$ *alors*

$$\forall\, t_1 < t_2 \in [0, T^*), \qquad \Big|\mathrm{dist}\,\big(x_i(t_2), \partial\Omega\big) - \mathrm{dist}\,\big(x_i(t_1), \partial\Omega\big)\Big| \ \le\ C\sqrt{t_2 - t_1}\,.$$

En particulier, la distance entre $x_i(t)$ *et* $\partial\Omega$ *tend vers 0.*

La première variation par rapport au cas du plan est que l'on peut distinguer les indices $i \notin I$ des points-vortex qui ne s'approchent pas du bord, puisque $d_0 > 0$, et $i \in I$ des points dont la distance au bord tend vers 0, excluant ainsi les comportements trop pathologique. Cependant, pour ces derniers, seule la régularité Hölderienne de la distance au bord est prouvée. Pour les points qui restent loin du bord, le comportement est similaire au cas du plan.

La preuve de ce résultat commence donc par effectuer cette séparation. Pour cela, on étudie la convergence

$$P_t := \sum_{i=1}^{N} a_i \, \delta_{x_i(t)} \longrightarrow \sum_{i=1}^{N} a_i \, b_i \, \delta_{x_i^*}, \qquad \text{as } t \to T^*$$

faiblement au sens des mesures sur Ω, où

$$b_i := \begin{cases} 0 & \text{if } x_i^* \in \partial\Omega, \\ 1 & \text{if } x_i^* \in \Omega. \end{cases}$$

Pour étudier la régularité de la distance au bord pour les indices $i \in I$, on établit l'estimation

$$\forall\, t \in [T_\delta, T^*), \qquad \left| \frac{\mathrm{d}}{\mathrm{d}t} \Big(\sum_{i \in J} a_i \Big)^{-1} \sum_{i \in J} a_i \, \mathrm{dist}\left(x_i(t); \partial\Omega\right) \right|$$
$$\leq \sum_{i \in J} \sum_{j \in I \setminus J} \frac{C_0}{|\mathrm{dist}\left(x_i(t), \partial\Omega\right) - \mathrm{dist}\left(x_j(t), \partial\Omega\right)|} + \sum_{i \in J} \frac{C_1}{\mathrm{dist}\left(x_i(t); \partial\Omega\right)} + C_2.$$

En posant $\zeta_i = \mathrm{dist}\,(x_i, \partial\Omega)$, on en déduit alors que la dynamique des ζ_i satisfait

$$\forall P \in \mathcal{P}(N), \qquad \left| \frac{\mathrm{d}}{\mathrm{d}t} B_P(t) \right| \leq \sum_{i \in P} \sum_{j \notin P} \frac{C_0}{|\zeta_i(t) - \zeta_j(t)|} + \sum_{i \in P} \frac{C_1}{|\zeta_i(t)|} + C_2.$$

Nous pouvons alors reprendre un raisonnement par clusterisation pour prouver la régularité Hölderienne des ζ_i. Cette dynamique est plus singulière, à cause de la présence des termes $\frac{1}{|\zeta_i(t)|}$. En particulier, 0, qui joue le rôle du bord dans la vraie dynamique des x_i, est un point important de cette dynamique.

Il est alors important d'étudier l'éventuelle présence d'un cluster proche de 0. Si un tel cluster existe, il n'est plus possible de contrôler la vitesse de son centre de vorticité comme nous l'avions fait dans le cas du plan. Nous ne sommes donc plus capables de prouver que chaque nouvelle clusterisation est une sous partition *stricte* de la précédente, et donc la finitude du nombre d'étape de notre construction.

Cependant, nous prouvons que le seul cas où la clusterisation n'est pas stricte, est le cas où le cluster proche de 0 s'est éloigné de 0 sans se briser. Or, par les mêmes arguments que dans le cas du plan, les clusters loin de 0 ne peuvent pas se rapprocher de 0 dans un temps trop court. Ainsi, si le cluster proche de 0 s'en éloigne, il n'y a plus de points proche de 0 jusqu'à T^*. Ce phénomène ne peut donc se produire qu'une seule fois, rajoutant au plus une unique étape à notre raisonnement.

2.4 Perspectives

2.4.1 Collision au bord du domaine

On s'intéresse au système point-vortex (1.20). Remarquons qu'à notre connaissance il n'existe aucun exemple de collision de point-vortex au bord du domaine, c'est-à-dire satisfaisant l'existence d'un indice i et d'un temps T^* tel que

$$\inf_{t \in [0, T^*[} d(x_i, \partial\Omega) = 0.$$

Au théorème 2.3.3, on a prouvé que dans les domaines bornés, ceci est équivalent à

$$\lim_{t\to T^*} d(x_i, \partial\Omega) = 0.$$

Paradoxalement, contrairement au plan, on ne connaît pas non plus de façon simple de s'assurer qu'il n'y ait pas de collision au bord. En effet, dans le plan, la conservation du Hamiltonien implique que si par exemple toutes les intensités sont de même signe, alors il n'y a pas de collision, et la solution est globale. Ce raisonnement ne marche plus dans un domaine borné parce que les termes singuliers sont de signe différent.

Présentons l'intuition suivante. Par la méthode des points-miroir, le système point-vortex dans le demi-plan peut s'interpréter comme le système point-vortex dans le plan entier : il suffit pour cela d'introduire aux positions symétriques des points-vortex, des points-vortex d'intensités opposées. Ainsi, une collision au bord du demi plan s'interprète comme une collision *neutre*, c'est à dire dont la somme des intensités est nulle. Or, une telle collision dans le plan ne peut pas être *mono-échelle*, c'est à dire toutes les distances impliquées sont en tout temps du même ordre de grandeur. Ceci est expliqué prouvé dans [32] par un argument énergétique. Ainsi on ne peut pas obtenir de collision mono-échelle au bord du demi plan.

Puisque dans n'importe quel domaine borné suffisamment régulier, au voisinage du bord, celui-ci s'approxime toujours par un bord plat, ceci nous donne l'intuition que les collisions au bord ne peuvent-être que dégénérées.

Puisqu'il est en quelque sorte "fondamental", étudions le cas le demi-plan $\mathbb{R}^2_+ = \{(x_1,x_2) \in \mathbb{R}^2, x_2 > 0\}$ plus en détail. Sa fonction de Green est explicite et s'écrit

$$G_{\mathbb{R}^2_+}(x,y) = \frac{1}{2\pi}\ln\frac{|x-y|}{|x-\bar{y}|}$$

où conformément aux notations complexes, $\bar{y} = (y_1, -y_2)$. Cette fonction de Green satisfait la relation d'antisymétrie :

$$\nabla_x^\perp G_{\mathbb{R}^2_+}(x,y)\cdot e_2 = -\nabla_x^\perp G_{\mathbb{R}^2_+}(y,x)\cdot e_2.$$

De plus,

$$\nabla^\perp \widetilde{\gamma}_{\mathbb{R}^2_+}(x)\cdot e_2 = 0.$$

Ceci nous pousse donc à mener l'étude suivante. Soit I l'ensemble des points collisionnant au bord. On a

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i\in I} a_i x_i(t)\cdot e_2\right) &= \sum_{\substack{i\in I\\ 1\le j\le N\\ j\neq i}} a_i a_j \nabla_x^\perp G_{\mathbb{R}^2_+}(x_i(t),x_j(t))\cdot e_2 + \frac{1}{2}\sum_{i\in I} a_i^2 \nabla^\perp\widetilde{\gamma}_{\mathbb{R}^2_+}(x_i(t))\cdot e_2\\ &= \sum_{\substack{i\in I\\ j\notin I}} a_i a_j \nabla_x^\perp G_{\mathbb{R}^2_+}(x_i(t),x_j(t))\cdot e_2\\ &= \sum_{\substack{i\in I\\ j\notin I}} \frac{a_i a_j}{2\pi}\left(\frac{(x_i(t)-x_j(t))\cdot e_1}{|x_i(t)-x_j(t)|^2} - \frac{(x_i(t)-\overline{x_j}(t))\cdot e_1}{|x_i(t)-\overline{x_j}(t)|^2}\right)\\ &= \sum_{\substack{i\in I\\ j\notin I}} \frac{a_i a_j}{2\pi}(x_i(t)-x_j(t))\cdot e_1\left(\frac{|x_i(t)-\overline{x_j}(t)|^2 - |x_i(t)-x_j(t)|^2}{|x_i(t)-x_j(t)|^2|x_i(t)-\overline{x_j}(t)|^2}\right),\end{aligned}$$

ce qui donne finalement que

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i\in I} a_i x_i(t)\cdot e_2\right) = \sum_{\substack{i\in I\\ j\notin I}} \frac{a_i a_j}{2\pi}(x_i(t)-x_j(t))\cdot e_1\left(\frac{4(x_i(t)\cdot e_2)(x_j(t)\cdot e_2)}{|x_i(t)-x_j(t)|^2|x_i(t)-\overline{x_j}(t)|^2}\right). \tag{2.10}$$

On note M le vecteur de vorticité

$$M(t) = \sum_{i \in I} a_i x_i(t).$$

Il existe deux cas dans lesquels on sait conclure à l'impossibilité des collisions.

Cas 1 : si tous les points collisionnent au bord, c'est-à-dire $I = \{1, \ldots, N\}$ alors on a

$$\frac{\mathrm{d}}{\mathrm{d}t} M(t) \cdot e_2 = 0.$$

Ainsi, si

$$M(0) \cdot e_2 = \sum_{i \in I} a_i x_i((0) \cdot e_2 \neq 0,$$

alors $M(T^*) \cdot e_2 \neq 0$, ce qui est absurde : I étant l'ensemble des indices des points-vortex collisionnant au bord, on a nécessairement $M(t) \cdot e_2 \underset{t \to T^*}{\longrightarrow} 0$.

Cas 2 : On suppose cette fois que toutes les intensités a_i sont positives. Si l'on sait affirmer que les point $j \notin I$ satisfont $d(x_j(t), \partial\Omega) \geq \delta)$ et que d'autre part on sait que les points I qui collisionnent au bord tendent vers le bord, comme détaillé au Chapitre 5 pour les domaines bornés [1], alors on a que $d(x_i(t), \partial\Omega) \leq \delta/2)$ à partir d'un certain temps t_0. Notons pour simplifier pour tout indice $i \in [\![1, N]\!]$,

$$d_i(t) = d(x_i(t), \partial\Omega).$$

Pour les temps $t \geq t_0$, pour $i \in I$ et $j \notin I$, on a $|x_i(t) - x_j(t)| \leq \delta/2$ donc

$$\left| (x_i(t) - x_j(t)) \cdot e_1 \left(\frac{4(x_i(t) \cdot e_2)(x_j(t) \cdot e_2)}{|x_i(t) - x_j(t)|^2 |x_i(t) - \overline{x_j}(t)|^2} \right) \right|$$
$$= \frac{4 d_i(t)}{|x_i(t) - x_j(t)|^2} \frac{\left|(x_i(t) - x_j(t)) \cdot e_1\right| \left(x_j(t) \cdot e_2\right)}{|x_i(t) - \overline{x_j}(t)|^2} \leq \frac{16}{\delta^2} d_i(t).$$

Reportant ceci dans l'inéquation (2.10) donne :

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} M(t) \cdot e_2 \right| \leq \sum_{\substack{i \in I \\ j \notin I}} \frac{8|a_i|\,|a_j|}{\pi \delta^2} d_i(t) \leq C \sum_{i \in I} a_i d_i(t) = C M(t) \cdot e_2$$

puisque les intensités sont positives. On peut alors appliquer le Lemme de Gronwall pour obtenir

$$M(t) \cdot e_2 \geq \exp\left(-Ct\right) M(0) \cdot e_2 > 0.$$

Et donc il existe au moins un point d'indice dans I qui ne collissionne pas au bord, ce qui est absurde par définition de I. Il n'y a donc aucun point collisionnant au bord, dans le demi plan, si toutes les masses sont positives.

En conclusion, on a obtenu plusieurs cas élémentaires où les collisions sont impossibles au bord du demi plan. Ceci est une piste de recherche intéressante : on peut se demander si les collisions au bord du demi plan sont possibles ou non, et bien entendu, essayer de généraliser aux domaines bornés généraux.

1. Dans le demi-plan, la preuve exposée ne fonctionne plus. En revanche, on peut utiliser directement la Proposition 5.3.6, puisque la dynamique des distances vérifie l'hypothèse (5.3.25) pour tout temps et pas simplement proche du bord. Ceci prouve la séparation des points dans le demi-plan.

2.4.2 Optimalité des bornes de confinement

Nous avons évoqué en section 2.1.2 la possibilité d'existence de point stationnaire instable pour la dynamique d'un unique point-vortex dans un domaine borné. Ces points instables permettent d'émettre la conjecture suivante à propos du problème de confinement : la borne logarithmique dans le Théorème 2.1.1 est optimale.

D'abord, dans un domaine bi-convexe bien choisi, on doit pouvoir construire des points stationnaires instable. Ensuite, nous le détaillerons au Chapitre 3, introduisons un temps de sortie pour le système point-vortex

$$\tau_{\varepsilon,\beta} = \sup\left\{t \geq 0, \forall s \in [0,t], |z(t) - x_0| \leq \varepsilon^\beta\right\}$$

en supposant $|z(0) - x_0| \leq \varepsilon$. C'est une formulation du temps de sortie que nous avons introduit dans le problème de confinement où cette fois le tourbillon est supposé singulier. Alors ce temps de sortie-ci satisfait l'existence d'une constante ξ_1 telle que pour tout $\varepsilon > 0$ assez petit,

$$\tau_{\varepsilon,\beta} < \xi_1 |\ln \varepsilon|.$$

On peut formuler ce résultat en disant que la borne logarithmique est optimale *pour les points-vortex*, pour le problème de confinement. Bien entendu, un tel tourbillon singulier ne rentre pas dans le contexte du problème de confinement. L'idée serait alors d'approcher cette donnée singulière par un tourbillon lisse pour conclure à l'optimalité de la borne logarithmique dans le Théorème 2.1.1.

Si le plan paraît clair en utilisant un point instable dans un domaine borné, on peut se demander alors si la borne est toujours optimale dans le plan entier. Sans bord, est-il encore possible de construire une configuration de sorte que l'un des points se comporte comme un point instable ? La configuration $z_1 = 0$, $z_2 = (1, 0)$ et $z_3 = (-1, 0)$, avec les intensités $a_1 = 1$, $a_2 = -2$ et $a_3 = -2$, fait évoluer le premier point dans un champ extérieur engendrant une dynamique instable, au moins à l'instant initial. L'avantage de cette construction est qu'elle est envisageable dans le contexte des équations SQG et peut donc également montrer l'optimalité du problème de confinement de l'article [18]. En revanche, le champ extérieur varie dès que les points se mettent en mouvement, et le contrôle de celui-ci peut s'en trouver plus délicat.

2.4.3 Confinement pour les configurations linéairement stables

En section 2.1.3, nous avons montré au Théorème 2.1.4 que la borne de confinement en puissance négative de ε peut être obtenue dans un domaine borné lorsque le confinement a lieu autour d'un unique point super-stable. Bien que nous ayons montré l'existence de domaines possédant de tels points, ce n'est pas le cas de tous les domaines. Toute ellipse qui n'est pas un cercle par exemple, n'en possède pas, nous le montrerons au Chapitre 3.

Il est donc très naturel de se poser la question suivante : la super-stabilité est elle vraiment nécessaire pour obtenir la borne de confinement améliorée ? Cette fois, étudier le problème de confinement pour un point-vortex, comme en section 2.4.2, ne nous est pas d'une grande aide car pour les points stables, tout comme les points super-stables, $\tau_{\varepsilon,\beta} = +\infty$, puisqu'un point-vortex placé à distance ε du point stationnaire restera à distance $C\varepsilon$ de celui-ci.

Ce qui nous empêche actuellement d'obtenir le résultat pour un point stable, c'est qu'il faut à la fois obtenir les estimations de contrôle du centre de vorticité et des moments dans la version forte que nous avons évoqué en section 2.1.3 tout en préservant la symétrie du problème pour annuler les singularités. À ce jour, sans super-stabilité, il ne nous est pas possible d'obtenir les deux. Néanmoins, rien ne nous indique non plus que cette super-stabilité soit réellement nécessaire.

Un cas particulièrement intéressant où la borne de confinement puissance pourrait être envisagé est les *vortex crystals*, voir [5]. Il s'agit des configurations de points-vortex qui se déplacent dans un mouvement *rigide*, c'est-à-dire que toutes les distances $t \mapsto |z_i(t) - z_j(t)|$ sont constantes. Ces configurations sont un cas particulier de configuration auto-similaire. L'étude de la stabilité linéaire de ces configurations remonte à [76]. Un résultat, présenté par exemple dans [3] et [5], est que les configurations polygonales de points-vortex identiques sont linéairement stable pour $N \leq 6$, et linéairement instable pour $N \geq 8$. Notons d'ailleurs que dans le contexte plus général des équations SQG, le problème de désingularisation autour de ces configuration a été traité dans [35]. En rajoutant un point-vortex au centre du polygone, d'intensité éventuellement différente, on peut même construire [14] une configuration stable et stationnaire pour $N \in [\![4, 14]\!]$.

Il serait donc pertinent d'étudier le problème de confinement pour ce genre de configuration, et il est raisonnable d'envisager que la borne de confinement puisse être en puissance de ε.

Chapitre 3

Long time confinement around a stable point

This chapter is constituted of the following published article :

Long time confinement of vorticity around a stable stationary point vortex in a bounded planar domain
With **Dragoș Iftimie**, Ann. Inst. H. Poincaré Anal. Non Linéaire, 2021, 38 (5), pp. 1461–1485.

In this paper we consider the incompressible Euler equation in a simply-connected bounded planar domain. We study the confinement of the vorticity around a stationary point vortex. We show that the power law confinement around the center of the unit disk obtained in [13] remains true in the case of a stationary point vortex in a simply-connected bounded domain. The domain and the stationary point vortex must satisfy a condition expressed in terms of the conformal mapping from the domain to the unit disk. Explicit examples are discussed at the end.

3.1 Introduction and main result

To study the behavior of an incompressible inviscid fluid, we consider the planar Euler equations in a bounded domain $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} \partial_t u(x,t) + u(x,t)\cdot\nabla u(x,t) = -\nabla p(x,t), & \forall (x,t) \in \Omega \times \mathbb{R}_+^* \\ u(x,0) = u_0(x), & \forall x \in \Omega \\ \nabla \cdot u(x,t) = 0, & \forall (x,t) \in \Omega \times \mathbb{R}_+ \\ u(x,t)\cdot \vec{n} = 0, & \forall (x,t) \in \partial\Omega \times \mathbb{R}_+ \end{cases} \tag{3.1.1}$$

where u denotes the velocity of the fluid, p its internal pressure, and n is the exterior normal to $\partial\Omega$. If $\Omega = \mathbb{R}^2$ the boundary condition should be changed into a vanishing condition at infinity. We define the fluid's vorticity by $\omega = \partial_1 u_2 - \partial_2 u_1$, which satisfies the equation :

$$\partial_t \omega(x,t) + u(x,t)\cdot\nabla\omega(x,t) = 0. \tag{3.1.2}$$

If the domain Ω is smooth, then we have a unique global smooth solution of (3.1.1), see [80]. In addition, if $\partial\Omega \in C^{1,1}$ we have the following result due to Yudovitch (see [81]) : for any $\omega_0 \in L^1 \cap L^\infty$, there exists a unique solution to (3.1.1), with $u \in L^\infty(\mathbb{R}^+, W^{1,p})$, for every $1 < p < \infty$, and $\omega \in L^\infty(\mathbb{R}^+, L^1 \cap L^\infty)$. The result is true in more general domains, in particular in domains with finite number of corners with angle strictly less than π and when the vorticity is compactly supported in Ω, see [56] and [41]. In particular, one could take Ω to be a convex polygon.

Let us denote by G_Ω the Green's function of Ω. Since the domain is supposed to be simply-connected, the velocity can be recovered from the vorticity through the following Biot-Savart law

$$u(x,t) = \int_\Omega \nabla_x^\perp G_\Omega(x,y)\omega(y,t)\mathrm{d}y, \tag{3.1.3}$$

where $x^\perp = (-x_2, x_1)$.

The point vortex system is a simplified version of the Euler equations where the vorticity is assumed to be a finite sum of Dirac masses $\omega_0 = \sum_{i=1}^N a_i \delta_{z_i}$. It was introduced by Helmholtz in [43], see also [62]. Since (3.1.2) is a transport equation, one expects the vorticity to remain a sum of Dirac masses at some points $z_i(t)$. The Biot-Savart law (3.1.3) reads in this case

$$u(x,t) = \sum_{i=1}^N a_i \nabla_x^\perp G_\Omega(x, z_i(t)).$$

However, if x is one of the points $z_i(t)$, then this velocity is not defined as $G_\Omega(x,y)$ is singular at $y = x$. But as x approaches $z_i(t)$, the singular part of the velocity defined above is given by fast rotation around that point. More precisely, since the map $G_\Omega - G_{\mathbb{R}^2}$ is harmonic in both its variable on Ω, the function $\gamma_\Omega : \Omega \times \Omega \to \mathbb{R}$, $\gamma_\Omega = G_\Omega - G_{\mathbb{R}^2} = G_\Omega - \frac{1}{2\pi}\ln|x-y|$ is smooth. So the singular part of $\nabla_x^\perp G_\Omega(x, z_i(t))$ is given by $\frac{(x-z_i)^\perp}{2\pi|x-z_i|^2}$. The point vortex system consists in ignoring this singular part which should have no influence on the motion of z_i itself. Denoting by $\widetilde{\gamma}_\Omega(x) = \gamma(x,x)$ the Robin function of the domain Ω we obtain then the following point vortex dynamic :

$$\forall 1 \le i \le N, \quad \frac{\mathrm{d}z_i(t)}{\mathrm{d}t} = \sum_{\substack{j=1 \\ j\neq i}}^N a_j \nabla_x^\perp G_\Omega(z_i(t), z_j(t)) + a_i \frac{1}{2}\nabla^\perp \widetilde{\gamma}_\Omega(z_i(t)), \tag{3.1.4}$$

where the $(a_i)_{1\le i\le N} \in \mathbb{R}\backslash\{0\}$ are the masses of the point vortices $(z_i(t))_{1\le i\le N}$. Equations (3.1.4) are called Kirchhoff-Routh equations.

Due to the singularities of the Green's function, these equations are valid only while the points $z_i(t)$ stay distinct and do not leave the domain Ω. There exist configurations leading to collapse of the point vortices, but they are exceptional, see [62] for the case of $\mathbb{R}^2$, and [61] for the case of the unit disk (we will discuss this more in detail later).

An important question in fluid dynamics is whether the point vortex system is a good approximation of the Euler equations. There are convergence results in both ways.

Let us first mention that the point vortex system was used as a numerical approximation of the Euler system. More precisely, consider a smooth solution of the Euler equations and construct an initial discrete vorticity which is a sum of Dirac masses located on a grid $(hj)_{j\in\mathbb{Z}^2}$ where $h \in \mathbb{R}$ is the length of the grid, with masses $h^2\omega_0(hj)$. Solve then the point vortex system with this initial vorticity. In [36], the authors proved that this point vortex method is consistent, stable, and converges to the smooth solution of the Euler equation.

The convergence from the Euler system to the point vortex system was also proved in [63]. More precisely, consider a smooth initial vorticity that is sharply concentrated around some initial point vortices z_i : the support of ω_0 is included in the union of the disks of radius ε around the points z_i, with $\varepsilon > 0$ being small. The authors of [63] proved that for any time τ, and for any $\delta > 0$, if $\varepsilon = \varepsilon(\tau, \delta) > 0$ is small enough then the solution stays sharply concentrated within disks of radius δ from the points $z_i(t)$ up to time τ. This statement can be seen as a fixed time confinement result.

In this paper we are interested in the so-called *long time confinement problem*, that is we want to know for how long confinement around point vortices remains true. More precisely, we want to understand how ε, δ and τ are linked together and we wish to obtain a confinement time

τ as large as possible. We already know from [63] that τ goes to infinity when ε goes to 0, but we would like to obtain an explicit rate as good as possible.

This problem was already studied by Buttà and Marchioro [13]. These authors assumed that $\delta = \varepsilon^\beta$, with $\beta < 1/2$, and made the following assumptions on the initial vorticity. Assume that $\omega_0 \in L^1 \cap L^\infty$ and there exists ν such that

$$\begin{cases} |\omega_0| \leq \varepsilon^{-\nu} \\ \omega_0 = \displaystyle\sum_{i=1}^N \omega_{0,i}, \;\; \operatorname{supp} \omega_{0,i} \subset D(z_i, \varepsilon) \\ \omega_{0,i} \text{ has a definite sign} \\ \displaystyle\int_\Omega \omega_{0,i} \mathrm{d}x = a_i. \end{cases} \tag{3.1.5}$$

Let $\omega(x,t)$ the solution of (3.1.2). We denote by $\tau_{\varepsilon,\beta}$ the exit time of the vorticity from the disks of radius ε^β :

$$\tau_{\varepsilon,\beta} = \sup \left\{ t \geq 0, \forall s \in [0,t], \operatorname{supp} \omega(\cdot, s) \subset \bigcup_{i=1}^N D(z_i(s), \varepsilon^\beta) \right\}. \tag{3.1.6}$$

For any N-tuple of distinct points $(z_i) \in \Omega$, there exists ε small enough, such that the disks $D(z_i(0), \varepsilon^\beta)$ are disjoints, and therefore this exit time is well defined and strictly positive. The aim is to obtain a lower bound on $\tau_{\varepsilon,\beta}$ depending explicitly on ε. Two results have been obtained in [13]. The first is a logarithmic confinement for the whole plane.

Théorème 3.1.1 ([13])**.** *Assume that $\Omega = \mathbb{R}^2$, that the initial vorticity satisfies* (3.1.5) *and that the point vortex system with initial data $\sum_{i=1}^n a_i \delta_{z_i}$ has a global solution. Then for every $\beta < 1/2$ there exists $\varepsilon_0 > 0$ and $C > 0$ such that*

$$\forall \varepsilon < \varepsilon_0, \;\; \tau_{\varepsilon,\beta} > C|\ln(\varepsilon)|.$$

The second result is more restrictive, it holds true for the unit disk and for a single point vortex located at the center, but the conclusion is much stronger since it gives a power-law confinement.

Théorème 3.1.2 ([13])**.** *Let $\Omega = D$ and ω_0 satisfying* (3.1.5) *with $N = 1$ and $z_1 = 0$, so that it is compactly supported within the disk $D(0, \varepsilon)$. Then for every $\beta < 1/2$ there exists $\varepsilon_0 > 0$ and $\alpha > 0$ such that :*

$$\forall \varepsilon < \varepsilon_0, \;\; \tau_{\varepsilon,\beta} > \varepsilon^{-\alpha}.$$

The aim of this paper is to extend Theorem 3.1.2 to more general domains. We also observe that Theorem 3.1.1 can also be extended to bounded domains ; we will discuss this problem in a forthcoming paper.

We consider a single point vortex in a simply-connected bounded domain. We assume for simplicity that the mass of the point vortex is 1, but the results below hold true for a general mass. The first question that arises is the location of the point vortex. We will show that there are special points that allow us to obtain the power-law lower bound while for others the logarithmic bound is probably optimal. The dynamic of a single point vortex of mass a reduces to the following ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) = a \frac{1}{2} \nabla^\perp \widetilde{\gamma}_\Omega(z(t)).$$

It is obvious from this ODE that a point vortex is stationary if and only if it is a critical point of the Robin function $\widetilde{\gamma}$. The Robin function has been studied, see [40], and we know that such critical points always exist in a bounded domain. Let x_0 be a critical point of the Robin function which is fixed for the rest of this paper. From the Riemann mapping theorem we know that there exists a biholomorphic map T from Ω to the unit disk D. We can chose T such that it maps x_0 to $0 : T(x_0) = 0$. We shall see below that x_0 is a critical point of the Robin function if and only if $T''(x_0) = 0$ (see Proposition 3.2.4). This condition therefore characterizes the fact that x_0 is a stationary point for the point vortex system. We call such points stationary points.

Our main result is the following.

Théorème 3.1.3. *Let Ω be a simply connected bounded domain of $\mathbb{R}^2$ with $C^{1,1}$ boundary. Let x_0 be a stationary point such that $T'''(x_0) = 0$ where T is a biholomorphism from Ω to the unit disk, mapping x_0 to 0. Assume that ω_0 satisfies* (3.1.5) *with $N = 1$ and $z_1 = x_0$. Then for every $\beta < 1/2$ and for any $\alpha < \min(\beta, 2 - 4\beta)$, there exists $\varepsilon_0 > 0$ such that*

$$\forall \varepsilon < \varepsilon_0, \;\; \tau_{\varepsilon,\beta} > \varepsilon^{-\alpha}.$$

This extends Theorem 3.1.2 to more general bounded domains. Indeed, in the case of the unit disk we can choose $T(z) = z$ so the hypothesis given above is verified for the center of the disk. Let us also observe that the hypothesis that x_0 is stationary induces no restriction on the domain Ω. Indeed, we recall that Gustafsson [40] proved that every simply-connected smooth domain has at least a stationary point. However, the hypothesis that $T'''(x_0) = 0$ is a condition that not all domains satisfy. We will comment on this in the last section. We will see in particular that any domain which is invariant by some rotation of angle $\theta \in (0, \pi)$ around x_0 satisfies the condition $T'''(x_0) = 0$.

In order to understand better the significance of the condition $T'''(x_0) = 0$, one could assume that the vorticity ω itself is a point vortex. We study in detail this perturbation problem in Section 3.3. We will prove there that if $|T'''(x_0)| < 2|T'(x_0)|^3$ then $\tau_{\varepsilon,\beta} = \infty$ if ε is small enough while if $|T'''(x_0)| > 2|T'(x_0)|^3$ then $\tau_{\varepsilon,\beta}$ is in general not better than $C|\ln \varepsilon|$, see Theorem 3.3.1. In other words, in this particular case we have long time confinement better than $C|\ln \varepsilon|$ if and only if $|T'''(x_0)| < 2|T'(x_0)|^3$. However, when ω is smooth we require the stronger assumption $T'''(x_0) = 0$.

The plan of the paper is the following. In Section 3.2 we introduce some notation and discuss some facts about the Green's function and the point vortex system. In section 3.3 we consider the particular case when ω is a point vortex itself. In Section 3.4 we prove Theorem 3.1.3. The last section contains some final remarks and some examples of domains for which our theorem applies.

3.2 Preliminary tools

List of notation :

- Ω is a $C^{1,1}$ bounded and simply connected domain of $\mathbb{R}^2$;
- $D(x_0, r)$ is the disk of center x_0 and of radius r and $D = D(0, 1)$;
- u is the velocity of the fluid and p its pressure, satisfying equations (3.1.1) ;
- $\omega = \partial_1 u_2 - \partial_2 u_1$ is the vorticity of the fluid ;
- supp f is the support of the function f, namely the closure of the set $\{x \in \Omega, f(x) \neq 0\}$;
- δ_z is the Dirac mass in z ;
- G_Ω or G is the Green's function of the domain Ω ;
- γ_Ω or γ is the regular part of G_Ω, see relation (3.2.1) ;
- $\widetilde{\gamma}_\Omega(x) = \gamma_\Omega(x, x)$ is the Robin function ;
- $C, C_1, C_2, \ldots ; K, K_1, K_2, \ldots, L$, are strictly positive constants that may vary from one line to another, when their value is not important to the result ;
- $a \cdot b$ is the scalar product of vectors in $\mathbb{R}^2$;
- ∇f, $D^2 f$ and $\nabla \cdot g$ are respectively the gradient of f, its Hessian matrix, and the divergence of g.

3.2.1 Green's Function

We recall that the Green's function of a domain Ω is the solution of

$$\Delta_x G_\Omega(x,y) = \delta(x-y)$$

vanishing at the boundary, and at infinity if Ω is unbounded. It is a symmetric function on Ω^2 that satisfies for $x \neq y$

$$\Delta_x (G_\Omega(x,y) - G_{\mathbb{R}^2}(x,y)) = 0,$$

which means that $G_\Omega - G_{\mathbb{R}^2}$ is a function, denoted by γ_Ω, which is harmonic in both of its variable. Therefore, we have :

$$G_\Omega(x,y) = \frac{1}{2\pi} \ln|x-y| + \gamma_\Omega(x,y) \tag{3.2.1}$$

where γ_Ω is symmetric and smooth.

In the particular case of $\Omega = D$, we have that

$$G_D(x,y) = \frac{\ln|x-y|}{2\pi} - \frac{\ln|x-y^*||y|}{2\pi} \tag{3.2.2}$$

where $y^* = \dfrac{y}{|y|^2}$ is the inverse relative to the unit circle. In particular,

$$\gamma_D(x,y) = -\frac{\ln|x-y^*||y|}{2\pi}. \tag{3.2.3}$$

Let $x_0 \in \Omega$. From the Riemann Mapping Theorem (see for instance [1], chapter 6) and recalling that Ω is simply-connected, we know that there exists a biholomorphic map T from Ω to the unit disk D. The map T is unique up to compositions with the biholomorphisms of the unit disk which are given by

$$\phi_{z_0,\lambda}(z) = \lambda \frac{z_0 - z}{1 - \overline{z_0} z},$$

with $z_0 \in D$ and $|\lambda| = 1$ arbitrary. Choosing z_0 and λ conveniently, we can assume without loss of generality that $T(x_0) = 0$ and that $T'(x_0)$ is a strictly positive real number. These two conditions insure the uniqueness of the conformal map T. Let us observe that in Theorem 3.1.3 the mapping T is not unique since we only assume that it maps x_0 to 0. However, the condition $T'''(x_0) = 0$ does not depend on the choice of T (once we assumed that it maps x_0 to 0). Indeed, if T_1 and T_2 are two biholomorphisms from Ω to D mapping x_0 to 0, then $T_1 \circ T_2^{-1}$ is a biholomorphism from D to D mapping 0 to 0. So it must be a rotation : there exists some λ of modulus 1 such that $T_1 \circ T_2^{-1}(z) = \lambda z$. So $T_1 = \lambda T_2$ and therefore $T_1'''(x_0) = \lambda T_2'''(x_0)$. Then $T_1'''(x_0) = 0$ if and only if $T_2'''(x_0) = 0$.

We will assume from now on that $T(x_0) = 0$. The assumption that $T'(x_0) > 0$ can be made but is not necessary. In the following, $T'(x_0)$ is a complex number.

The properties of the conformal map T imply a precise description of the Green's function of Ω. Indeed, a Green's function composed with a conformal mapping is another Green's function, see for example [1] chapter 6. Therefore the formula (3.2.2) yields the following proposition.

Proposition 3.2.1. *Let T be the biholomorphic mapping introduced above. Then*

$$G_\Omega(x,y) = G_D(T(x),T(y)) = \frac{\ln|T(x)-T(y)|}{2\pi} - \frac{\ln|T(x)-T(y)^*||T(y)^*|}{2\pi}.$$

In the following, we will have to use both characterizations of the map T, as a $\mathbb{C} \to \mathbb{C}$ map, and as a $\mathbb{R}^2 \to \mathbb{R}^2$ map. In particular, we recall that $T'(x) = \partial_1 T(x) = \partial_1 T_1(x) + i\partial_1 T_2(x)$, and that $\partial_1 T_1 = \partial_2 T_2$ and $\partial_2 T_1 = -\partial_1 T_2$. So for any map $f \in C^1(\mathbb{R}^2, \mathbb{R})$, we have that

$$\nabla(f \circ T)(x) = \begin{pmatrix} \partial_1 T_1(x)\partial_1 f(T(x)) + \partial_1 T_2(x)\partial_2 f(T(x)) \\ -\partial_1 T_2(x)\partial_1 f(T(x)) + \partial_1 T_1(x)\partial_2 f(T(x)) \end{pmatrix},$$

or as complex numbers :

$$\begin{aligned}\partial_1(f\circ T)(x)+i\partial_2(f\circ T)(x) &= \mathrm{Re}(T'(x))\partial_1 f(T(x))+\mathrm{Im}(T'(x))\partial_2 f(T(x)) \\ &\qquad + i[-\mathrm{Im}(T'(x))\partial_1 f(T(x))+\mathrm{Re}(T'(x))\partial_2 f(T(x))] \\ &= \overline{T'(x)}(\partial_1 f(T(x))+i\partial_2 f(T(x))).\end{aligned}$$

Identifying $\nabla f=\partial_1 f+i\partial_2 f$, we have :

$$\nabla(f\circ T)(x)=\overline{T'(x)}\nabla f(T(x)), \tag{3.2.4}$$

where the product above must be understood as the product of two complex numbers. We will frequently use this property in the following.

From Proposition 3.2.1 and from relation (3.2.1) applied for Ω and for D we get that

$$\gamma_\Omega(x,y)=\gamma_D(T(x),T(y))+\frac{1}{2\pi}\ln\frac{|T(x)-T(y)|}{|x-y|},$$

and letting $y\to x$ we obtain at the level of Robin functions

$$\forall x\in\Omega,\quad \widetilde{\gamma}_\Omega(x)=\widetilde{\gamma}_D(T(x))+\frac{1}{2\pi}\ln|T'(x)|. \tag{3.2.5}$$

Using the explicit expression of γ_D, see relation (3.2.3), we can compute the gradient of γ_Ω :

$$\begin{aligned}\nabla_x\gamma_\Omega(x,y) &= \frac{\overline{T'}(x)(T(x)-T(y))}{2\pi|T(x)-T(y)|^2}-\frac{\overline{T'}(x)(T(x)-T(y)^*)}{2\pi|T(x)-T(y)^*|^2}-\frac{(x-y)}{2\pi|x-y|^2} \\ &= \frac{\overline{T'(x)}}{2\pi\overline{(T(x)-T(y))}}-\frac{\overline{T'(x)}}{2\pi\overline{(T(x)-T(y)^*)}}-\frac{1}{2\pi\overline{(x-y)}}\end{aligned} \tag{3.2.6}$$

where we used relation (3.2.4).

The mapping $\widetilde{\gamma}_\Omega:\Omega\to\mathbb{R}^2$, that will be named only $\widetilde{\gamma}$ when there is no ambiguity, has been studied extensively in [40]. In particular, it was shown in that paper that $-\widetilde{\gamma}_\Omega$ is a super-harmonic function, that is that $\Delta\widetilde{\gamma}_\Omega>0$, and that it goes to infinity near the boundary like $-\frac{1}{2\pi}\ln(d(x,\partial\Omega))$. This implies that we have the following proposition (see [40]).

Proposition 3.2.2 ([40]). *For every bounded and simply connected open set Ω, there exists at least one point x_0 where $\widetilde{\gamma}$ reaches its minimum.*

Critical points of $\widetilde{\gamma}_\Omega$ will be of special interest in the following. The proposition above implies the existence of a critical point of the Robin function. Moreover, if Ω is convex, then the critical point is also unique (see [40]). Though we will not use this result here, it may be interesting to keep this in mind, especially when looking for explicit examples.

3.2.2 Point vortex system

The point vortex dynamic, which is described by equations (3.1.4), can exhibit finite time blow-up of solutions. One scenario of blow-up is when two point vortices hit each other in finite time, meaning that there exists $i\neq j$ and $t<\infty$ such that $z_i(t)=z_j(t)$. This phenomena can happen, see for instance [62] or [50] for an example of finite-time collapse of a self similar evolution of point vortices. Another scenario for blow-up is when a point vortex hits the boundary. However, the finite time blow-up is exceptional in the case of the whole plane, see [62], and for the unit disk, see [61]. In those cases, the N-dimensional Lebesgue measure of the set of initial positions that lead to a collapse is 0. The case of a more general bounded domain is an ongoing work.

Let us recall the convergence theorem obtained in [63], for configurations of point vortices that do not lead to finite-time blow-up :

Théorème 3.2.3 ([63]). *Let $(a_i, z_i(t))$ be a global solution of the point vortex system* (3.1.4). *For every $\delta > 0$ and for every time τ, there exists $\varepsilon > 0$ such that if ω_0 satisfies* (3.1.5) *for some $0 < \nu < 8/3$, then the vorticity stays confined up to the time τ in disks of radius δ centered on $z_i(t)$.*

In addition, the authors of [15] proved that $\tau_{\varepsilon,\beta} \to +\infty$ as ε goes to 0 for any $\beta < 1/3$ (recall that $\tau_{\varepsilon,\beta}$ was defined in relation (3.1.6)). However, these theorems don't say anything about how δ depends on the time τ, or conversely, for how long this confinement remains true, depending on ε.

For some explicit examples of point vortices we refer to [46].

As mentioned in the introduction, the stationary point vortices are the critical points of the Robin function. Indeed, the relation (3.1.4) with $N = 1$ reduces to

$$z'(t) = \frac{a}{2}\nabla^{\perp}\widetilde{\gamma}_{\Omega}(z(t)).$$

We refer to such a critical point x_0 by saying that it is a *stationary* point vortex.

Those points can also be characterized in terms of the conformal mapping T as zeroes of T''.

Proposition 3.2.4. *The following conditions are equivalent :*
(i) A single point vortex placed in x_0 is stationary,
(ii) $\nabla\widetilde{\gamma}_{\Omega}(x_0) = 0$,
(iii) $T''(x_0) = 0$.

Démonstration. This proposition was already proved in [40]. We recall the proof for the convenience of the reader.

We already observed that (*i*) and (*ii*) are equivalent. We only need to prove that (*ii*) is equivalent to (*iii*). From relations (3.2.5) and (3.2.4), and recalling that T' can't vanish in Ω, we deduce that

$$\nabla\widetilde{\gamma}_{\Omega}(x) = \nabla\widetilde{\gamma}_{D}(T(x))\overline{T'(x)} + \frac{1}{2\pi}\frac{T'(x)\overline{T''(x)}}{|T'(x)|^2}.$$

One can easily check that 0 is a stationary point for the unit disk, so $\nabla\widetilde{\gamma}_D(0) = 0$. Thus

$$\nabla\widetilde{\gamma}_{\Omega}(x_0) = \frac{1}{2\pi}\overline{\left(\frac{T''(x_0)}{T'(x_0)}\right)}.$$

Clearly $\nabla\widetilde{\gamma}_{\Omega}(x_0) = 0$ is equivalent to $T''(x_0) = 0$. □

3.3 Confinement for Dirac mass around a stationary vortex

The aim of this section is to study the case where the vorticity ω itself is a Dirac mass. The reason we consider this simpler case is because it is easier to have a complete description of what happens. And this in turn gives us an indication of what to expect in the smooth case considered in Theorem 3.1.3.

We consider a single point vortex located at $z(t)$ which is close to the stationary point vortex x_0. Rescaling time if needed, we can assume without loss of generality that the mass of the point-vortex is 1. The question of knowing if $z(t)$ remains close to x_0 is closely related to the notion of *stability* of x_0 that we discuss in what follows.

3.3.1 Stability of a stationary point vortex

Since the mass of the point vortex $z(t)$ is 1, its equation of motion is the following

$$z'(t) = \frac{1}{2}\nabla^{\perp}\widetilde{\gamma}_{\Omega}(z(t)). \tag{3.3.1}$$

We have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\gamma}_\Omega(z(t)) = z'(t) \cdot \nabla\widetilde{\gamma}_\Omega(z(t)) = 0$$

which means that the point vortex is evolving on the level set $\widetilde{\gamma}(x) = \text{cst}$. We know from [40] that the Robin function $\widetilde{\gamma}(x)$ goes to infinity as x approaches the boundary of Ω. Therefore, the point vortex $z(t)$ can never reach the boundary so (3.3.1) has a global solution.

Since $-\widetilde{\gamma}_\Omega$ is super-harmonic, the eigenvalues of the real symmetric matrix $D^2\widetilde{\gamma}_\Omega(x_0)$ have positive sum, meaning that at least one of them is positive. So two main cases can occur : either both eigenvalues are positive, either one is positive and one is negative. We skip the study of the degenerate case when one eigenvalue is 0.

So let $\lambda_+ > 0$ and λ_- be the two eigenvalues of $D^2\widetilde{\gamma}_\Omega(x_0)$. If $\lambda_- > 0$, the Morse Lemma implies that there exists a change of variables $y = \phi(x)$ in the neighborhood of x_0 such that in this neighborhood of x_0,

$$\widetilde{\gamma}_\Omega(x) = \widetilde{\gamma}_\Omega(\phi^{-1}(y)) = \widetilde{\gamma}_\Omega(x_0) + (y - x_0)_1^2 + (y - x_0)_2^2.$$

In particular, in these new local coordinates, the level sets of $\widetilde{\gamma}_\Omega$ are circles, so in the real coordinates, they are diffeomorphic to circles. More precisely, the level set $\widetilde{\gamma}_\Omega(\phi^{-1}(y)) = \widetilde{\gamma}_\Omega(x_0) + r$ is a circle of radius $\sqrt{r}$, provided that $r > 0$ is small enough. Because ϕ is a C^∞ diffeomorphism, and $\phi(x_0) = x_0$, there exist constants $k, K > 0$ such that $k|y - x_0| < |x - x_0| < K|y - x_0|$ and thus the level set $\widetilde{\gamma}_\Omega(x) = \widetilde{\gamma}_\Omega(x_0) + r$ is contained in the annulus $\{k\sqrt{r} < |x - x_0| < K\sqrt{r}\}$.

We conclude from these observations that if $|z(0) - x_0| \leq \varepsilon$ with ε small enough, then we have that $|z(t) - x_0| \leq \varepsilon\frac{K}{k}$ for every time $t \geq 0$. This a stability property : if ε is small enough and if we assume that the point vortex starts at distance ε of x_0, then it remains at distance of order ε for any time.

Assume now that $\lambda_- < 0$. We have in this case that

$$\widetilde{\gamma}_\Omega(x) = \widetilde{\gamma}_\Omega(\phi^{-1}(y)) = \widetilde{\gamma}_\Omega(x_0) - (y - x_0)_1^2 + (y - x_0)_2^2,$$

so the level sets in a neighborhood of x_0 are in the local coordinates y hyperbolas, with the special level set $\widetilde{\gamma}_\Omega(x) = \widetilde{\gamma}_\Omega(x_0)$ being a union of two line segments. In particular, we will see later that one line segment is repulsive, meaning that when a point vortex evolves on that segment, it moves away from the point x_0. We call this situation unstable. In that case, no matter how close the point vortex starts from the critical point x_0, it goes away from x_0 exponentially fast, as we will see in section 3.3.3.

Let us summarize what we observed above. If the eigenvalues of the matrix $D^2\widetilde{\gamma}_\Omega(x_0)$ are both positive, x_0 is said to be stable, and a point vortex close enough to x_0 remains indefinitely close to it. If one of the eigenvalues is negative, then x_0 is said to be unstable, because there exists a point vortex evolving on a level set going away from x_0. This notion of stability is the same as the one introduced in [62], Chapter 3. Figure 3.1 shows examples of the two situations.

3.3.2 Taylor expansion near a stationary vortex

The aim is now to express the stability of a stationary point vortex in terms of the conformal map T. We already know from Proposition 3.2.4 that x_0 is stationary if and only if $T''(x_0) = 0$. We will now make a Taylor expansion of T around x_0 in order to have a better understanding of the point vortex dynamic near a critical point. We therefore need to compute the Hessian matrix $D^2\widetilde{\gamma}_\Omega$ in that point. By relation (3.2.5) we have that

$$\frac{1}{2}D^2\widetilde{\gamma}_\Omega(x_0) = \frac{1}{2}D^2(\widetilde{\gamma}_D \circ T)(x_0) + \frac{1}{4\pi}D^2(\ln|T'|)(x_0).$$

This expression is actually explicit in terms of the conformal map T ; indeed, we know that $\widetilde{\gamma}_D(x) = -\frac{1}{2\pi}\ln(1 - |x|^2)$. We recall that $T''(x_0) = 0$, so $\partial_1 T'(x_0) = \partial_2 T'(x_0) = 0$. We recall

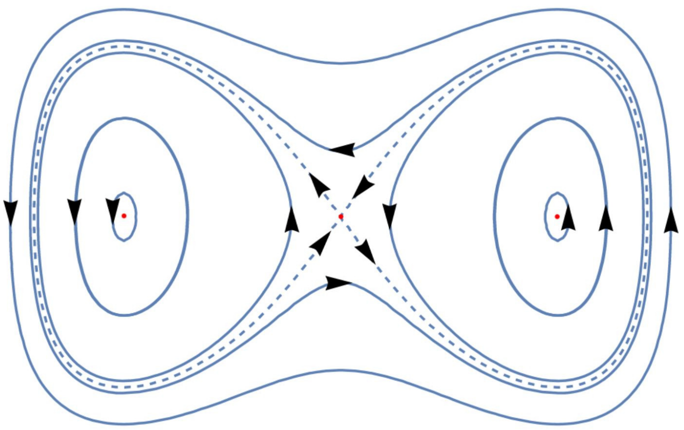


FIGURE 3.1 – Trajectories of a single point vortex in a domain with two stable points and an unstable one.

also that for an holomorphic function φ we have that $\partial_1\varphi = \varphi'$ and $\partial_2\varphi = i\varphi'$. Some direct computations give that

$$D^2(\widetilde{\gamma}_D \circ T)(x_0) = \frac{1}{\pi}\begin{pmatrix}|T'(x_0)|^2 & 0 \\ 0 & |T'(x_0)|^2\end{pmatrix}$$

and

$$\partial_i\partial_j(\ln|T'|)(x_0) = \frac{\partial_i\partial_j T'(x_0)\cdot T'(x_0)}{|T'(x_0)|^2}$$

so that

$$D^2(\ln|T'|)(x_0) = \frac{1}{|T'(x_0)|^2}\begin{pmatrix} T'''(x_0)\cdot T'(x_0) & (T'''(x_0))^{\perp}\cdot T'(x_0) \\ (T'''(x_0))^{\perp}\cdot T'(x_0) & -T'''(x_0)\cdot T'(x_0)\end{pmatrix}.$$

We conclude that

$$\frac{1}{2}D^2\widetilde{\gamma}_\Omega(x_0) = \frac{1}{4\pi}\begin{pmatrix}2\mu^2+p & q \\ q & 2\mu^2-p\end{pmatrix}, \tag{3.3.2}$$

where

$$\begin{cases}\mu^2 = |T'(x_0)|^2 \\ p = \dfrac{1}{\mu^2}T'''(x_0)\cdot T'(x_0) \\ q = \dfrac{1}{\mu^2}(T'''(x_0))^{\perp}\cdot T'(x_0).\end{cases}$$

We compute the characteristic polynomial of the matrix $\frac{1}{2}D^2\widetilde{\gamma}_\Omega(x_0)$:

$$\det(\frac{1}{2}D^2\widetilde{\gamma}_\Omega(x_0) - XI_2) = X^2 - \frac{\mu^2}{\pi}X + \frac{1}{16\pi^2}\left(4\mu^4 - p^2 - q^2\right)$$

and the eigenvalues are

$$\lambda_\pm = \frac{2\mu^2 \pm \sqrt{p^2+q^2}}{4\pi}.$$

Furthermore, we notice that $p^2+q^2 = \dfrac{|T'''(x_0)|^2}{|T'(x_0)|^2}$. We deduce from the expression of the determinant that x_0 is stable when $\lambda_- > 0$, that is when $2|T'(x_0)|^3 > |T'''(x_0)|$ and unstable when $2|T'(x_0)|^3 < |T'''(x_0)|$.

Another interesting thing to notice is that when $T'''(x_0) = 0$, the matrix $D^2\widetilde{\gamma}_\Omega(x_0) = \frac{\mu^2}{2\pi}I_2$, with I_2 the identity matrix, and thus the trajectories of the linearized system $z'(t) = (D^2\widetilde{\gamma}_\Omega(x_0)(z(t)-x_0))^{\perp}$ are circles of center x_0.

3.3.3 Exit time around an unstable stationary point

We now consider the unstable case, meaning $\lambda_+\lambda_- < 0$. We now go back to (3.3.1). Using relation (3.3.2) and the subsequent calculations, one can check that the eigenvalues of the Jacobian matrix of $\frac{1}{2}\nabla^\perp\widetilde{\gamma}_\Omega$ in x_0 are $\xi = \sqrt{-\lambda_+\lambda_-}$ and $-\xi$.

We thus know from [42] Theorem 6.1 of Chapter 9, and the corollary and remarks associated to this theorem that there exist two local invariant manifolds for the equation, and thus there exist two special solutions to equation (3.3.1) associated with each eigenvalue $\pm\xi$. We denote by z_1 the solution that goes away from x_0, corresponding to the eigenvalue ξ. It satisfies that

$$\forall 0 < c < \xi, \quad \lim_{t\to-\infty} |z_1(t) - x_0| e^{-ct} = 0. \tag{3.3.3}$$

We denote by t_1 the first time when $|z_1(t_1) - x_0| = \varepsilon^\beta$ and by t_2 the last time when $|z_1(t_2) - x_0| = \varepsilon$. Provided ε is small enough, we have that $0 > t_1 > t_2$. We define $\widetilde{z}(t) = z_1(t + t_2)$. This is a solution of equation (3.3.1) such that $|\widetilde{z}(0) - x_0| = \varepsilon$ and thus we can set

$$\tau_{\varepsilon,\beta} = \max\left\{t \geq 0, \forall s \in [0,t], |\widetilde{z}(t) - x_0| \leq \varepsilon^\beta\right\}$$

which is the exit time as defined in relation 3.1.6 associated to a point vortex starting at distance ε of x_0 and evolving along the same trajectory as z_1. By construction, it satisfies that $\tau_{\varepsilon,\beta} = t_1 - t_2 \leq -t_2$.

We also know from relation (3.3.3) that there exist a constant M and a constant $0 < c < \xi$, such that for every $t < 0$, we have that $|z_1(t) - x_0|e^{-ct} \leq M$. Therefore,

$$\varepsilon = |z_1(t_2) - x_0| \leq M e^{ct_2}.$$

Thus, $\ln\varepsilon \leq \ln M + ct_2$ which means that $\tau_{\varepsilon,\beta} \leq -t_2 \leq \frac{2}{c}|\ln\varepsilon|$ for ε small enough.

This result empowers the conjecture that one cannot expect any general confinement result, as in Theorem 3.1.1, better than $C|\ln\varepsilon|$, since we have an example of such an exit time in the case where the vorticity is itself a point vortex.

We can summarize the results of this section in the following theorem :

Théorème 3.3.1. *Assume that $T(x_0) = T''(x_0) = 0$, so that x_0 is a stationary point. If $|T'''(x_0)| < 2|T'(x_0)|^3$ then any single point vortex starting close to x_0 will remain indefinitely close to it. More precisely, there exists some small constant $\varepsilon_0 > 0$ such that if $\varepsilon \leq \varepsilon_0$ and $z(0) \in D(x_0, \varepsilon)$ then $z(t) \in D(x_0, C\varepsilon)$ for all times t and $\tau_{\varepsilon,\beta} = +\infty$.*

Conversely, the condition $|T'''(x_0)| > 2|T'(x_0)|^3$ insures that for every $\varepsilon > 0$, there exists a starting position $z(0)$ such that $|z(0) - x_0| \leq \varepsilon$, but the point vortex exits the disk $D(x_0, \varepsilon^\beta)$ with $\beta < 1$ in finite time $\tau_{\varepsilon,\beta} \leq C|\ln\varepsilon|$.

3.4 Power-law confinement near a stationary vortex

The present section is devoted to the proof of Theorem 3.1.3. To simplify the notation, we denote from now on $\gamma = \gamma_\Omega$, $\widetilde{\gamma} = \widetilde{\gamma}_\Omega$ and $G = G_\Omega$.

Recall that Ω is a simply connected and bounded domain, that T is a biholomorphism from Ω to D such that $T(x_0) = 0$. We have that x_0 is a stationary point which, according to Proposition 3.2.4, means that $T''(x_0) = 0$.

We assume that ω_0 satisfies (3.1.5) with $N = 1$ and $z_1 = x_0$. Without loss of generality we can assume that the mass a_1 is 1 (changing the mass means rescaling the time). We therefore have that ω_0 is non negative, is supported in $D(x_0, \varepsilon)$ and $\|\omega\|_{L^1} = 1$.

Let us introduce

$$F(x,t) = \int \nabla_x^\perp \gamma(x,y)\omega(y,t)\mathrm{d}y, \tag{3.4.1}$$

which is the influence of the blob over itself through the boundary. The key point of the proof of Theorem 3.1.2 from [13] in the case of the disk is the following property :

$$\forall x, z \in D(0,\delta), \forall t \geq 0, \quad |F(x,t) - F(z,t)| \leq C|x-z|\delta^2. \tag{3.4.2}$$

To prove that the result remains true for more general domains, we will need a similar inequality. It is important to notice that unlike the case when ω is a point vortex itself, we don't necessarily have that $F(x_0,t) = 0$ *a priori.* According to (3.2.6) we have that

$$2\pi\overline{\nabla_x\gamma(x_0,y)} = -\frac{T'(x_0)}{T(y)} + \frac{T'(x_0)}{T(y)^*} - \frac{1}{x_0 - y}.$$

Recalling that $T(y)^* = 1/\overline{T(y)}$ and making the Taylor expansion

$$T(y) = T'(x_0)(y-x_0) + \frac{T'''(x_0)}{6}(y-x_0)^3 + \mathcal{O}(|y-x_0|^4)$$

we get that

$$\begin{aligned}
2\pi\overline{\nabla_x\gamma(x_0,y)} &= -\frac{1}{y-x_0}\frac{1}{1+\frac{T'''(x_0)}{6T'(x_0)}(y-x_0)^2 + \mathcal{O}(|y-x_0|^3)} + T'(x_0)\overline{T}(y) - \frac{1}{x_0-y} \\
&= -\frac{1}{y-x_0}\left(1 - \frac{T'''(x_0)}{6T'(x_0)}(y-x_0)^2 + \mathcal{O}(|y-x_0|^3)\right) \\
&\qquad + T'(x_0)\overline{T'(x_0)(y-x_0)} + \mathcal{O}(|y-x_0|^2) - \frac{1}{x_0-y}.
\end{aligned}$$

We conclude that

$$2\pi\nabla_x\gamma(x_0,y) = \overline{\frac{T'''(x_0)}{6T'(x_0)}(y-x_0)} + |T'(x_0)|^2(y-x_0) + \mathcal{O}(|y-x_0|^2). \tag{3.4.3}$$

3.4.1 Estimate of the influence of the boundary

The first step of the proof is a technical lemma giving a Lipschitz inequality for F. This is the counterpart in Ω of the relation (3.4.2) valid for the unit disk.

Lemme 3.4.1. *We have the following estimate, for δ sufficiently small and for every $x, y, z \in D(x_0,\delta)$*

$$\overline{\nabla_x\gamma(x,y)} - \overline{\nabla_x\gamma(z,y)} = (x-z)\left(\frac{T'''(x_0)}{6\pi T'(x_0)} + \mathcal{O}(\delta)\right), \tag{3.4.4}$$

where the term $\mathcal{O}(\delta)$ is bounded by $C\delta$ with C a constant depending only on Ω and x_0. In particular, if we assume that $T'''(x_0) = 0$, there exists a constant $K_1 = K_1(\Omega, x_0)$ such that

$$\forall x, y, z \in D(x_0,\delta), \quad |\nabla_x\gamma(x,y) - \nabla_x\gamma(z,y)| \leq K_1|x-z|\delta. \tag{3.4.5}$$

Furthermore, if we assume that $\operatorname{supp}\omega \subset D(x_0,\delta)$ *we have that*

$$\forall x, z \in D(x_0,\delta), \quad |F(x,t) - F(z,t)| \leq K_1|x-z|\delta. \tag{3.4.6}$$

Démonstration. Let us define

$$R(x,y,z) = 2\pi(\overline{\nabla_x\gamma(x,y) - \nabla_x\gamma(z,y)}).$$

Using (3.2.6) we can write

$$R(x,y,z) = \frac{T'(x)}{T(x)-T(y)} - \frac{T'(x)}{T(x)-T(y)^*} - \frac{1}{x-y} - \left(\frac{T'(z)}{T(z)-T(y)} - \frac{T'(z)}{T(z)-T(y)^*} - \frac{1}{z-y}\right)$$

so

$$R(x,y,z) = \frac{T'(x)(T(z)-T(y)) - T'(z)(T(x)-T(y))}{(T(x)-T(y))(T(z)-T(y))} + \frac{x-z}{(x-y)(z-y)} - \frac{T'(x)(T(z)-T(y)^*) - T'(z)(T(x)-T(y)^*)}{(T(x)-T(y)^*)(T(z)-T(y)^*)}.$$

We decompose R into two parts :

$$R = R_1 - R_2$$

with

$$R_1(x,y,z) = \frac{T'(x)(T(z)-T(y)) - T'(z)(T(x)-T(y))}{(T(x)-T(y))(T(z)-T(y))} + \frac{x-z}{(x-y)(z-y)}$$

and

$$\begin{aligned} R_2(x,y,z) &= \frac{T'(x)(T(z)-T(y)^*) - T'(z)(T(x)-T(y)^*)}{(T(x)-T(y)^*)(T(z)-T(y)^*)} \\ &= \frac{T'(x)T(z) - T'(z)T(x)}{(T(x)-T(y)^*)(T(z)-T(y)^*)} + \frac{T(y)^*(T'(z)-T'(x))}{(T(x)-T(y)^*)(T(z)-T(y)^*)} \\ &\equiv R_{21} + R_{22}. \end{aligned}$$

Since T is smooth and $T(x_0) = 0$, we clearly have that $|T(x)| \le C\delta$ and $|T(y)| \le C\delta$. So we can estimate

$$|T(x) - T(y)^*| \ge |T(y)^*| - |T(x)| = \frac{1}{|T(y)|} - |T(x)| \ge \frac{1}{C\delta} - C\delta \ge \frac{1}{2C\delta}$$

if δ is sufficiently small. We can therefore bound R_{21} as follows :

$$|R_{21}| \le 4C^2\delta^2 |T'(x)T(z) - T'(z)T(x)| \le C\delta^2|x-z|.$$

Similarly

$$|T(x) - T(y)^*| \ge |T(y)^*| - |T(x)| = \frac{1}{|T(y)|} - |T(x)| \ge \frac{1}{2|T(y)|}$$

so

$$\frac{|T(y)^*|}{|T(x)-T(y)^*|} \le 2|T(y)||T(y)^*| = 2.$$

Then

$$|R_{22}| = \frac{|T(y)^*|}{|T(x)-T(y)^*|}\frac{|T'(z)-T'(x)|}{|T(z)-T(y)^*|} \le 4C\delta|T'(z)-T'(x)| \le C\delta|x-z|^2 \le C\delta^2|x-z|.$$

We conclude from the estimates above that

$$|R_2(x,y,z)| \le C\delta^2|x-z|. \tag{3.4.7}$$

To estimate R_1, we write it under the form

$$R_1(x,y,z) = \frac{N(x,y,z)}{(T(x)-T(y))(T(z)-T(y))(x-y)(z-y)}$$

with

$$N(x,y,z) = [T'(x)(T(z)-T(y)) - T'(z)(T(x)-T(y))](x-y)(z-y) + (x-z)(T(x)-T(y))(T(z)-T(y)). \tag{3.4.8}$$

As T is holomorphic we observe that N is holomorphic in the variables x, y et z. One can notice that N is 0 if $x = z$, so it can be factorized by $x - z$. Let us also recall that γ is smooth everywhere, so R is smooth too. We proved above that R_2 is bounded in $D(x_0, \delta)^3$, so $R_1 = R + R_2$ is also bounded in this set. But the denominator of R_1 has a factor $(x-y)^2(y-z)^2$, so it follows that $N(x, y, z)$ can be factorized by $(x - y)^2(y - z)^2$. But we observed that N can also be factorized by $x - z$. This implies that there exists a holomorphic function $N_1(x, y, z)$ such that :

$$N(x, y, z) = (x - y)^2(z - y)^2(x - z)N_1(x, y, z). \tag{3.4.9}$$

Therefore

$$\frac{R_1(x, y, z)}{x - z} = \frac{(x - y)(z - y)N_1(x, y, z)}{(T(x) - T(y))(T(z) - T(y))}. \tag{3.4.10}$$

We need now to compute $N_1(x_0, x_0, x_0)$. To do that, we will differentiate five times relation (3.4.9) and evaluate it in (x_0, x_0, x_0). It is clear that derivatives up to order 4 of $(x - y)^2(z - y)^2(x - z)$ all vanish at (x_0, x_0, x_0). We can therefore write that

$$\partial_x^3\partial_z^2 N(x_0, x_0, x_0) = \partial_x^3\partial_z^2\big[(x - y)^2(z - y)^2(x - z)\big](x_0, x_0, x_0)N_1(x_0, x_0, x_0) = 12N_1(x_0, x_0, x_0).$$

Differentiating relation (3.4.8) allows to find after some computations that

$$\partial_x^3\partial_z^2 N(x_0, x_0, x_0) = 4T'(x_0)T'''(x_0).$$

Therefore,

$$N_1(x_0, x_0, x_0) = \frac{T'(x_0)T'''(x_0)}{3}$$

so

$$N_1(x, y, z) = N_1(x_0, x_0, x_0) + \mathcal{O}(\delta) = \frac{T'(x_0)T'''(x_0)}{3} + \mathcal{O}(\delta).$$

We now go back to (3.4.10). We observe that

$$\frac{x - y}{T(x) - T(y)}$$

is smooth on $D(x_0, \delta)^3$ with value $1/T'(x)$ when $x = y$. Therefore

$$\frac{x - y}{T(x) - T(y)} = \frac{1}{T'(x_0)} + \mathcal{O}(\delta).$$

Similarly

$$\frac{z - y}{T(z) - T(y)} = \frac{1}{T'(x_0)} + \mathcal{O}(\delta).$$

Combining the previous relations results in

$$\frac{R_1(x, y, z)}{x - z} = \Big(\frac{1}{T'(x_0)} + \mathcal{O}(\delta)\Big)^2\Big(\frac{T'(x_0)T'''(x_0)}{3} + \mathcal{O}(\delta)\Big) = \frac{T'''(x_0)}{3T'(x_0)} + \mathcal{O}(\delta)$$

so

$$R_1(x, y, z) = (x - z)\left(\frac{T'''(x_0)}{3T'(x_0)} + \mathcal{O}(\delta)\right).$$

Recalling (3.4.7) finally implies that

$$R(x, y, z) = (x - z)\left(\frac{T'''(x_0)}{3T'(x_0)} + \mathcal{O}(\delta)\right),$$

which proves (3.4.4). Clearly (3.4.5) follows from (3.4.4) if $T'''(x_0) = 0$. Finally, relation (3.4.6) follows from (3.4.5) after integrating and recalling that the mass of ω is 1. □

Comparing (3.4.6) and (3.4.2), we see that we lose the factor δ^2. But in the case $T'''(x_0) = 0$, we still get a factor δ and this is enough to make our argument work. Actually the proof of [13] would still be correct assuming only that $|F(x,t) - F(z,t)| \le C|x - z|\delta$. In our proof, a factor δ^2 would improve the restriction over the power α in Theorem 3.1.3. However, please notice that if $T'''(x_0) \neq 0$ we lose the factor δ and our proof does not work anymore.

3.4.2 Estimates of the trajectories

From now on we assume that $T'''(x_0) = 0$.
Let us introduce the center of vorticity :

$$B(t) = \int_\Omega x\omega(x,t)\mathrm{d}x, \tag{3.4.11}$$

and the moment of inertia :

$$I(t) = \int_\Omega |x - B|^2 \omega(x,t)\mathrm{d}x.$$

For future needs, let us compute the time derivative of B. Recall that ω satisfies the equation (3.1.2) in the sense of distributions and that it is compactly supported. We have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}B(t) &= \int x\partial_t\omega(x,t)\mathrm{d}x \\
&= -\int xu(x,t)\cdot\nabla\omega(x,t)\mathrm{d}x \\
&= \int (u(x,t)\cdot\nabla)x\ \omega(x,t)\mathrm{d}x \\
&= \int u(x,t)\omega(x,t)\mathrm{d}x \\
&= \iint \left(\frac{(x-y)^\perp}{2\pi|x-y|^2} + \nabla_x^\perp\gamma(x,y)\right)\omega(y,t)\omega(x,t)\mathrm{d}x\mathrm{d}y
\end{aligned}$$

where we used (3.1.3) and (3.2.1).

Observing that $\frac{(x-y)^\perp}{2\pi|x-y|^2}$ is antisymmetric when exchanging x and y and recalling the definition of F given in (3.4.1), we infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}B(t) = \int F(x,t)\omega(x,t)\mathrm{d}x. \tag{3.4.12}$$

Let us define

$$R_t = \max\{|x - B(t)|;\ x \in \operatorname{supp}\omega(t)\}$$

and choose some $X(t) \in \operatorname{supp}\omega(t)$ such that $|X(t) - B(t)| = R_t$. We denote by $s \mapsto X_t(s)$ the trajectory passing through $X(t)$ at time t so that $X_t(t) = X(t)$.

We have the following lemma which allows us to control the time evolution of R_t :

Lemme 3.4.2. *For any $t \le \tau_{\varepsilon,\beta}$ we have that*

$$\frac{\mathrm{d}}{\mathrm{d}s}|X_t(s) - B(s)|\Big|_{s=t} \le 2K_1\varepsilon^\beta R_t + \frac{5}{\pi R_t^3}I(t) + K_2\left(\varepsilon^{-\nu}\int_{|x-B|>R_t/2}\omega(x,t)\mathrm{d}x\right)^{1/2}$$

where ν is the constant from relation (3.1.5)*, K_1 is the constant from Lemma 3.4.1 and K_2 is a universal constant.*

This lemma shows that in order to obtain upper bounds for the growth of the support of ω, one needs estimates for $I(t)$, $B(t)$ and for the mass of vorticity far from the center of mass.

Démonstration. We have that for any $s \ge 0$ and $t \ge 0$, $X_t'(s) = u(X_t(s), s)$, so

$$\frac{\mathrm{d}}{\mathrm{d}s}|X_t(s) - B(s)| = \big(u(X_t(s), s) - B'(s)\big)\cdot\frac{X_t(s) - B(s)}{|X_t(s) - B(s)|}.$$

We fix now the time $t \geq 0$, we take $s = t$, and we write X instead of $X_t(t)$. By the Biot-Savart law (3.1.3), the relation (3.4.1) and recalling that the vorticity is assumed to be of integral 1, we have that

$$u(X,t) = F(X,t) + \int \frac{(X-y)^\perp}{2\pi|X-y|^2}\omega(y,t)\mathrm{d}y = \int \big(F(X,t) + \frac{(X-y)^\perp}{2\pi|X-y|^2}\big)\omega(y,t)\mathrm{d}y.$$

Relation (3.4.12) now implies that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}|X_t(s) - B(s)|\big|_{s=t} &= \left[\int \left(F(X,t) + \frac{(X-y)^\perp}{2\pi|X-y|^2} - F(y,t)\right)\omega(y,t)\mathrm{d}y\right]\cdot\frac{X-B(t)}{|X-B(t)|}\\ &= H_1 + H_2\end{aligned}$$

where

$$H_1 = \left[\int \left(F(X,t) - F(y,t)\right)\omega(y,t)\mathrm{d}y\right]\cdot\frac{X-B(t)}{|X-B(t)|},$$

and

$$H_2 = \left[\int \frac{(X-y)^\perp}{2\pi|X-y|^2}\omega(y,t)\mathrm{d}y\right]\cdot\frac{X-B(t)}{|X-B(t)|},$$

Thanks to Lemma 3.4.1 and recalling that $t \leq \tau_{\varepsilon,\beta}$, we have that

$$\begin{aligned}|H_1| = \left|\int \left(F(X,t) - F(y,t)\right)\omega(y,t)\mathrm{d}y\right| &\leq K_1\varepsilon^\beta \int |X-y|\omega(y,t)\mathrm{d}y\\ &\leq K_1\varepsilon^\beta 2R_t\end{aligned}$$

where we used that $\operatorname{supp}\omega \subset \overline{D(B,R)}$. The second term H_2 is the same as the left hand side of relation (2.28) in [13], and its estimate is the same :

$$|H_2| \leq \frac{5}{\pi R_t^3}I(t) + \left(\frac{1}{\pi}\varepsilon^{-\nu}\int_{|x-B|>R_t/2}\omega(x,t)\mathrm{d}x\right)^{1/2}$$

The lemma is now proved. □

3.4.3 Estimates of the moments of the vorticity

We have the following lemma :

Lemme 3.4.3. *For every $t < \min(\tau_{\varepsilon,\beta}, \varepsilon^{-\beta})$, we have :*

$$I(t) \leq K_3\varepsilon^2.$$

and :

$$|B(t) - x_0| \leq K_3\varepsilon,$$

where K_3 is a positive constant depending only on Ω and x_0.

Démonstration. We differentiate $I(t)$:

$$I'(t) = \int \left(|x-B|^2\partial_t\omega(x,t) - 2B'(t)\cdot(x-B)\omega(x,t)\right)\mathrm{d}x = \int |x-B|^2\partial_t\omega(x,t)\mathrm{d}x$$

where we used relation (3.4.11). Next, we use the equation of ω given in (3.1.2) and write

$$\begin{aligned}
I'(t) &= \int \big(-|x-B|^2 u(x,t)\cdot\nabla\omega(x,t)\big)\,\mathrm{d}x \\
&= \int 2(x-B)\cdot u(x,t)\omega(x,t)\mathrm{d}x \\
&= \iint 2(x-B)\cdot\nabla_x^\perp G(x,y)\omega(x,t)\omega(y,t)\mathrm{d}x\mathrm{d}y \\
&= \iint 2(x-B)\cdot\left[\nabla_x^\perp\gamma(x,y)+\frac{(x-y)^\perp}{2\pi|x-y|^2}\right]\omega(x,t)\omega(y,t)\mathrm{d}x\mathrm{d}y \\
&= \iint 2(x-B)\cdot\nabla_x^\perp\gamma(x,y)\omega(x,t)\omega(y,t)\mathrm{d}x\mathrm{d}y \\
&\qquad + \iint \frac{-x\cdot y^\perp - B\cdot(x-y)^\perp}{\pi|x-y|^2}\omega(x,t)\omega(y,t)\mathrm{d}x\mathrm{d}y
\end{aligned}$$

Exchanging x and y shows that the last term above vanishes. So

$$\begin{aligned}
I'(t) &= \iint 2(x-B)\cdot\nabla_x^\perp\gamma(x,y)\omega(x,t)\omega(y,t)\mathrm{d}x\mathrm{d}y \\
&= \iint 2(x-B)\cdot[\nabla_x^\perp\gamma(x,y)-\nabla_x^\perp\gamma(B,y)]\omega(x,t)\omega(y,t)\mathrm{d}x\mathrm{d}y \\
&\le 2K_1\varepsilon^\beta \iint |x-B|^2\omega(x,t)\omega(y,t)\mathrm{d}x\mathrm{d}y \\
&= 2K_1\varepsilon^\beta I(t)
\end{aligned}$$

where we used (3.4.5) with $\delta=\varepsilon^\beta$ because $(x,y,z)\in B(x_0,\varepsilon^\beta)$ (see the definition (3.1.6) of $\tau_{\varepsilon,\beta}$). The Gronwall lemma now implies that $I(t)\le I(0)\exp(2K_1\varepsilon^\beta t)$. Since at the initial time we have that $\operatorname{supp}\omega_0$ and $B(0)$ are in the disk $D(x_0,\varepsilon)$ we infer that $I(0)\le 4\varepsilon^2$. We assumed that $t<\varepsilon^{-\beta}$ so we finally obtain that

$$\forall t\le\min(\tau_{\varepsilon,\beta},\varepsilon^{-\beta}),\quad I(t)\le 4e^{2K_1}\varepsilon^2.$$

We estimate now the center of vorticity. By relations (3.4.12) and (3.4.1), we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|B(t)-x_0|^2 &= 2B'(t)\cdot(B(t)-x_0) \\
&= 2\iint \nabla_x^\perp\gamma(x,y)\omega(y,t)\omega(x,t)\mathrm{d}x\mathrm{d}y\cdot(B(t)-x_0).
\end{aligned}$$

To estimate this, we use Lemma 3.4.1 and relation (3.4.3) and we recall that we assume that $T'''(x_0)=0$ and $\operatorname{supp}\omega\in D(x_0,\varepsilon^\beta)$ to obtain

$$\begin{aligned}
2\pi\nabla_x^\perp\gamma(x,y) &= 2\pi(\nabla_x^\perp\gamma(x,y)-\nabla_x^\perp\gamma(x_0,y)+\nabla_x^\perp\gamma(x_0,y)) \\
&= 2\pi(\nabla_x^\perp\gamma(x,y)-\nabla_x^\perp\gamma(x_0,y))+|T'(x_0)|^2(y-x_0)^\perp+\mathcal{O}\left(|y-x_0|^2\right) \\
&= |T'(x_0)|^2(y-x_0)^\perp+\varepsilon^\beta(|x-x_0|+|y-x_0|)\mathcal{O}(1).
\end{aligned}$$

Putting the estimates above together we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|B(t)-x_0|^2 = 2\iint &\left[\frac{|T'|^2(x_0)(y-x_0)^\perp}{2\pi}+\varepsilon^\beta(|x-x_0|+|y-x_0|)\mathcal{O}(1)\right] \\
&\qquad\qquad \omega(y,t)\omega(x,t)\mathrm{d}x\mathrm{d}y\cdot(B(t)-x_0).
\end{aligned}$$

But we have the following cancellation :

$$\begin{aligned}\iint |T'|^2(x_0)(y-x_0)^\perp\omega(y,t)\omega(x,t)\mathrm{d}x\mathrm{d}y\cdot(B(t)-x_0)\\ &= |T'|^2(x_0)\iint(y-x_0)^\perp\omega(y,t)\omega(x,t)\mathrm{d}x\mathrm{d}y\cdot(B(t)-x_0)\\ &= |T'|^2(x_0)(B(t)-x_0)^\perp\cdot(B(t)-x_0)\\ &= 0.\end{aligned}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}|B(t)-x_0|^2 \le C|B(t)-x_0|\varepsilon^\beta\int|x-x_0|\omega(x,t)\mathrm{d}x.$$

We notice now that

$$\begin{aligned}\int|x-x_0|^2\omega(x,t)\mathrm{d}x &= \int|x-B(t)|^2\omega(x,t)\mathrm{d}x\\ &\quad-\int(x_0-B(t))\cdot(x-x_0+x-B(t))\omega(x,t)\mathrm{d}x\\ &= \int|x-B(t)|^2\omega(x,t)\mathrm{d}x-(x_0-B(t))\cdot(B(t)-x_0+B(t)-B(t))\\ &= I(t)+|x_0-B(t)|^2.\end{aligned}$$

By the Cauchy-Schwarz inequality

$$\int|x-x_0|\omega(x,t)\mathrm{d}x \le \Big(\int|x-x_0|^2\omega(x,t)\mathrm{d}x\Big)^{\frac12} = (I+|B(t)-x_0|^2)^{\frac12},$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}|B(t)-x_0| \le C\varepsilon^\beta(I^{\frac12}+|B(t)-x_0|).$$

By the Gronwall lemma we infer that

$$|B(t)-x_0| \le \exp\left(C\varepsilon^\beta t\right)\left(|B(0)-x_0|+C\varepsilon^\beta\int_0^t I(s)^{\frac12}\mathrm{d}s\right).$$

We already know that $I(s)\le C\varepsilon^2$, and $|B(0)-x_0|\le\varepsilon$, so

$$|B(t)-x_0| \le \exp(C\varepsilon^\beta t)(\varepsilon+C\varepsilon^\beta t\varepsilon).$$

As we assumed that $\varepsilon^\beta t\le 1$ we conclude that

$$|B(t)-x_0|\le C\varepsilon.$$

This completes the proof of the lemma. □

We have just shown that up to the time $\min(\tau_{\varepsilon,\beta},\varepsilon^{-\beta})$ the center of mass B stays close to x_0 and the moment of inertia I remains small. We need a last technical lemma, which is inspired by the appendix of [51].

Lemme 3.4.4. *For every $k\ge1$ and $t\le\min(\tau_{\varepsilon,\beta},\varepsilon^{-\beta})$ there exists a small constant $\varepsilon_0=\varepsilon_0(k)$, a large constant $C(k)$ and a constant K_4 which depends only on Ω and x_0, such that if*

$$\varepsilon\le\varepsilon_0 \qquad \textit{and} \qquad r^4\ge K_4\varepsilon^2\left(1+kt\ln(2+t)\right),$$

then

$$\int_{|x-B|>r}\omega(x,t)\mathrm{d}x \le C(k)\frac{\varepsilon^{k/2}}{r^k}.$$

Démonstration. Let us introduce the moment of vorticity of order $4n$

$$m_n(t) = \int_\Omega |x - B(t)|^{4n}\omega(x,t)\mathrm{d}x.$$

One can differentiate to obtain, by using relations (3.1.3), (3.4.12) and recalling that $\nabla \cdot u = 0$

$$\begin{aligned} m_n'(t) &= -\int |x - B(t)|^{4n} u(x,t) \cdot \nabla\omega(x,t)\mathrm{d}x \\ &\qquad - 4n\int B'(t)\cdot(x-B(t))|x-B(t)|^{4n-2}\omega(x,t)\mathrm{d}x \\ &= 4n\int u(x,t)\cdot(x-B(t))|x-B(t)|^{4n-2}\omega(x,t)\mathrm{d}x \\ &\qquad - 4nB'(t)\cdot\int(x-B(t))|x-B(t)|^{4n-2}\omega(x,t)\mathrm{d}x \\ &= 4n\iint \nabla_x^\perp G(x,y)\cdot(x-B(t))|x-B(t)|^{4n-2}\omega(x,t)\omega(y,t)\mathrm{d}x\mathrm{d}y \\ &\quad - 4n\iint \nabla_x^\perp\gamma(z,y)\omega(z,t)\omega(y,t)\mathrm{d}z\mathrm{d}y\cdot\int(x-B(t))|x-B(t)|^{4n-2}\omega(x,t)\mathrm{d}x. \end{aligned}$$

Recalling that $G(x,y) = \frac{\ln|x-y|}{2\pi} + \gamma(x,y)$ and that $\widetilde{\omega}(x',t) = \omega(x'+B(t),t))$, we can further decompose

$$m_n'(t) = a_n(t) + b_n(t) - c_n(t)$$

where

$$a_n(t) = 4n\iint \frac{(x'-y')^\perp}{2\pi|x'-y'|^2}\cdot x'|x'|^{4n-2}\widetilde{\omega}(x',t)\widetilde{\omega}(y',t)\mathrm{d}y'\mathrm{d}x',$$

$$b_n(t) = 4n\iint \nabla_x^\perp\gamma(x,y)\cdot(x-B(t))|x-B(t)|^{4n-2}\omega(x,t)\omega(y,t)\mathrm{d}x\mathrm{d}y,$$

$$c_n(t) = 4n\iiint \nabla_x^\perp\gamma(z,y)\cdot(x-B(t))|x-B(t)|^{4n-2}\omega(x,t)\omega(y,t)\omega(z,t)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

We observe that a_n is exactly the same quantity that appears in [51] on the last line of page 1726. Observing that the center of mass of $\widetilde{\omega}(x,t)$ is in 0, we deduce that the estimates given in [51] are true for a_n. More precisely, a_n satisfies the estimate given on the line 5, page 1729 of [51] :

$$|a_n(t)| \le Cn^2 I(t) m_{n-1}(t).$$

Applying Lemma 3.4.3 we obtain that

$$|a_n(t)| \le Cn^2\varepsilon^2 m_{n-1}(t). \tag{3.4.13}$$

The terms b_n and c_n are similar. We decompose

$$\nabla_x^\perp\gamma(x,y) = \nabla_x^\perp\gamma(x,y) - \nabla_x^\perp\gamma(x_0,y) + \nabla_x^\perp\gamma(x_0,y).$$

We apply Lemma 3.4.1 with $\delta = \varepsilon^\beta$ to deduce that

$$|\nabla_x^\perp\gamma(x,y) - \nabla_x^\perp\gamma(x_0,y)| \le K_1|x-x_0|\varepsilon^\beta \le K_1\varepsilon^{2\beta}$$

for all $x, y \in \operatorname{supp}\omega(t)$. Moreover, we also have that $|x - B(t)| \le C\varepsilon^\beta$ at least for ε small enough, since $|x - x_0| \le \varepsilon^\beta$ and $|B(t) - x_0| \le K_3\varepsilon$. Using these estimates we infer that

$$|b_n(t)| \le Cn\varepsilon^{2\beta}\int |x-B(t)|^{4n-1}\omega(x,t)\mathrm{d}x + d_n \le Cn\varepsilon^{5\beta} m_{n-1}(t) + d_n$$

where
$$d_n = 4n \left| \iint \nabla_x^\perp \gamma(x_0, y) \cdot (x - B(t))|x - B(t)|^{4n-2} \omega(x,t)\omega(y,t) \mathrm{d}x\mathrm{d}y \right|.$$

Decomposing $\nabla_x^\perp \gamma(z, y) = \nabla_x^\perp \gamma(z, y) - \nabla_x^\perp \gamma(x_0, y) + \nabla_x^\perp \gamma(x_0, y)$ we obtain that the same estimate holds true for c_n :
$$|c_n(t)| \leq Cn\varepsilon^{5\beta} m_{n-1}(t) + d_n.$$

We estimate now d_n. From relation (3.4.3) we know that there exists a bounded function $c(y)$ such that $\nabla_x^\perp \gamma(x_0, y) = [a(y - x_0) + c(y)|y - x_0|^2]^\perp$ where $a = \frac{|T'(x_0)|^2}{2\pi}$. We thus have that

$$\begin{aligned}
d_n &= 4n \left| \iint [a(y - x_0) + c(y)|y - x_0|^2]^\perp \cdot (x - B(t))|x - B(t)|^{4n-2} \omega(x,t)\omega(y,t) \mathrm{d}x\mathrm{d}y \right| \\
&\leq 4na \left| \int (B(t) - x_0)^\perp \cdot (x - B(t))|x - B(t)|^{4n-2} \omega(x,t) \mathrm{d}x \right| \\
&\qquad\qquad + Cn \iint |y - x_0|^2 |x - B(t)|^{4n-1} \omega(x,t)\omega(y,t) \mathrm{d}x\mathrm{d}y \\
&\leq 4na \left| \int (B(t) - x_0)^\perp \cdot (x - B(t))|x - B(t)|^{4n-2} \omega(x,t)) \mathrm{d}x \right| + Cn\varepsilon^{5\beta} m_{n-1}(t) \\
&\leq Cn(|B(t) - x_0|\varepsilon^{3\beta} + \varepsilon^{5\beta}) m_{n-1}(t).
\end{aligned}$$

Let us recall that Lemma 3.4.3 gives that $|B(t) - x_0| \leq K_3\varepsilon$ and therefore
$$|b_n(t)| + |c_n(t)| \leq Cn\varepsilon^{5\beta} m_{n-1}(t) + 2d_n \leq Cn(\varepsilon^{1+3\beta} + \varepsilon^{5\beta}) m_{n-1}(t).$$

Together with relation (3.4.13) and recalling that $m_n'(t) = a_n(t) + b_n(t) - c_n(t)$, the last estimate yields that
$$|m_n'(t)| \leq Cn^2(\varepsilon^2 + \varepsilon^{1+3\beta} + \varepsilon^{5\beta}) m_{n-1}(t).$$

Let us observe now that it suffices to assume that $\beta > 2/5$. Indeed, assume that Theorem 3.1.3 is proved for any $\beta \in]2/5, 1/2[$. Let $\beta' \leq 2/5$ and $\alpha < \min(\beta', 2 - 4\beta')$. Since $\alpha < 2/5$, one can easily check that there exists $\beta \in]2/5, 1/2[$ such that $\alpha < \min(\beta, 2 - 4\beta)$ (one can choose β close to $2/5$). Since $\beta' < \beta$, we also have that $\tau_{\varepsilon,\beta} \leq \tau_{\varepsilon,\beta'}$ so $\tau_{\varepsilon,\beta'} > \varepsilon^{-\alpha}$.

We assume in the sequel that $\beta > 2/5$. Due to this additional assumption, we have that
$$|m_n'(t)| \leq Cn^2\varepsilon^2 m_{n-1}(t). \tag{3.4.14}$$

Using Hölder's inequality on $f(x) = \omega^{1/n}(x)$ and $g(x) = |x - B(t)|^{4n-4}\omega^{1-1/n}(x)$ with $p = n$ and $q = \frac{n}{n-1}$, we have that

$$\begin{aligned}
m_{n-1}(t) &= \int |x - B(t)|^{4n-4} \omega(x,t) \mathrm{d}x \\
&\leq \left(\int \omega(x,t) \mathrm{d}x \right)^{1/n} \left(\int |x - B(t)|^{4n} \omega(t,x) \mathrm{d}x \right)^{(n-1)/n} \\
&= m_n(t)^{(n-1)/n}.
\end{aligned}$$

This last inequality combined with relation (3.4.14) gives that
$$m_n'(t) \leq Cn^2\varepsilon^2 m_n(t)^{(n-1)/n}.$$

We integrate to obtain that
$$m_n(t) \leq \left(m_n(0)^{1/n} + Cn\varepsilon^2 t \right)^n.$$

Clearly, $m_n(0) \leq (2\varepsilon)^{4n}$ so, assuming that $\varepsilon \leq 1$,

$$m_n(t) \leq \left(L\varepsilon^2(1+nt)\right)^n$$

for some constant L which depends only on Ω and x_0. Let us choose any $k \geq 1$, r and n such that

$$r^4 \geq 2L\varepsilon^2\left(1 + k\frac{\ln(2+t)}{\ln 2}t\right)$$

and

$$k\frac{\ln(2+t)}{\ln 2} - 1 < n \leq k\frac{\ln(2+t)}{\ln 2}.$$

This defines $n \geq 1$ since $k \geq 1$ and $\frac{\ln(2+t)}{\ln 2} \geq 1$. It also implies that $2^{n+1} > (2+t)^k$. Thus we have that

$$\begin{aligned}
\int_{|x-B|>r} \omega(x,t)\mathrm{d}x &= \int_{|x-B|>r} \omega(x,t)\frac{|x-B(t)|^{4n}}{|x-B(t)|^{4n}}\mathrm{d}x \\
&\leq \frac{m_n(t)}{r^{4n}} \\
&\leq \frac{\left(L\varepsilon^2(1+nt)\right)^n}{r^k r^{4n-k}} \\
&\leq \frac{1}{r^k}\frac{\left(L\varepsilon^2(1+k\frac{\ln(2+t)}{\ln 2}t)\right)^n}{\left(2L\varepsilon^2(1+k\frac{\ln(2+t)}{\ln 2}t)\right)^{n-k/4}} \\
&= \frac{1}{r^k}\frac{\left(L\varepsilon^2(1+k\frac{\ln(2+t)}{\ln 2}t)\right)^{k/4}}{2^{n-k/4}} \\
&= \frac{2^{k/4+1}}{r^k}\varepsilon^{k/2}\frac{\left(L(1+k\frac{\ln(2+t)}{\ln 2}t)\right)^{k/4}}{2^{n+1}} \\
&\leq \frac{2^{k/4+1}}{r^k}\varepsilon^{k/2}\frac{\left(L(1+k\frac{\ln(2+t)}{\ln 2}t)\right)^{k/4}}{(2+t)^k}.
\end{aligned}$$

The function $t \mapsto \dfrac{\left(L(1+k\frac{\ln(2+t)}{\ln 2}t)\right)^{k/4}}{(2+t)^k}$ is bounded on $\mathbb{R}^+$ for every k, so there exists a constant $C(k)$ such that for ε small enough :

$$\int_{|x-B|>r} \omega(x,t)\mathrm{d}x \leq C(k)\frac{\varepsilon^{k/2}}{r^k}.$$

This completes the proof of the lemma. □

3.4.4 End of the proof of Theorem 3.1.3

We can now finish the proof of Theorem 3.1.3.

We recall that, according to Lemma 3.4.2, we have that for each particle such that $|X(t) - B(t)| = R_t$,

$$\frac{\mathrm{d}}{\mathrm{d}s}|X_t(s) - B(s)|\Big|_{s=t} \leq 2K_1\varepsilon^\beta R_t + \frac{5}{\pi R_t^3}I(t) + K_2\left(\varepsilon^{-\nu}\int_{|x-B|>R_t/2}\omega(x,t)\mathrm{d}x\right)^{1/2}.$$

Due to the estimates obtained in Lemma 3.4.3, taking $t < \min(\tau_{\varepsilon,\beta}, \varepsilon^{-\beta})$, we infer that

$$\frac{\mathrm{d}}{\mathrm{d}s}|X_t(s) - B(s)|\Big|_{s=t} \leq 2K_1\varepsilon^{\beta} R_t + \frac{5K_3\varepsilon^2}{\pi R_t^3} + K_2\left(\varepsilon^{-\nu}\int_{|x-B|>R_t/2} \omega(x,t)\mathrm{d}x\right)^{1/2}. \tag{3.4.15}$$

Let us introduce f the solution of the ODE :

$$\begin{cases} f'(t) = 4K_1\varepsilon^{\beta} f(t) + 4\max\left(\dfrac{5K_3\varepsilon^2}{\pi f^3(t)}, K_2\left(\varepsilon^{-\nu}\displaystyle\int_{|x-B|>f(t)/2} \omega(x,t)\mathrm{d}x\right)^{1/2}\right) \\ f(0) = 4\varepsilon. \end{cases} \tag{3.4.16}$$

We want to show that for every $t \in [0, \min(\tau_{\varepsilon,\beta}, \varepsilon^{-\beta})]$, $R_t < f(t)$. Assume that this assertion is false, and let t_2 be the first time when it breaks down. Since $f(0) = 4\varepsilon$ and $R_0 \leq 2\varepsilon$, we infer that $t_2 > 0$. Let $s \mapsto X_{t_2}(s)$ be a trajectory such that $|X_{t_2}(t_2) - B(t_2)| = R_{t_2} = f(t_2)$. From (3.4.15) and (3.4.16), we see that

$$\frac{\mathrm{d}}{\mathrm{d}s}|X_{t_2}(s) - B(s)|\Big|_{s=t_2} < f'(t_2). \tag{3.4.17}$$

However, we have that for every $0 < h < t_2$,

$$|X_{t_2}(t_2 - h) - B(t_2 - h)| \leq R_{t_2-h} < f(t_2 - h)$$

which implies that

$$\frac{|X_{t_2}(t_2-h) - B(t_2-h)| - |X_{t_2}(t_2) - B(t_2)|}{-h} > \frac{f(t_2-h) - f(t_2)}{-h}$$

since $|X_{t_2}(t_2) - B(t_2)| = R_{t_2} = f(t_2)$. Taking the limit as $h \to 0$, we get a contradiction with (3.4.17).

We choose now α and $k > 6$ such that

$$0 < \alpha < \min(\beta, 2 - 4\beta)$$

and

$$k(1/2 - \beta) + 6\beta - 4 - \nu > 0.$$

We define

$$t_2 = \inf\{t > 0, f(t) = \varepsilon^{\beta}\}$$

and

$$t_1 = \sup\{t < t_2, f(t) = \varepsilon^{\beta}/2\}$$

so that $t_1 < t_2$ and

$$f(t_1) = \frac{\varepsilon^{\beta}}{2}, \quad f(t_2) = \varepsilon^{\beta} \quad \text{and} \quad \frac{\varepsilon^{\beta}}{2} \leq f(t) \leq \varepsilon^{\beta} \quad \forall t \in [t_1, t_2].$$

If $t_2 \geq \varepsilon^{-\alpha}$ then $R_t < f(t) \leq \varepsilon^{\beta}$ for all $t \in [0, \min(\tau_{\varepsilon,\beta}, \varepsilon^{-\beta}, \varepsilon^{-\alpha})] = [0, \min(\tau_{\varepsilon,\beta}, \varepsilon^{-\alpha})]$. By definition of $\tau_{\varepsilon,\beta}$ we know that $R_{\tau_{\varepsilon,\beta}} = \varepsilon^{\beta}$. So necessarily $\tau_{\varepsilon,\beta} \geq \varepsilon^{-\alpha}$ which completes the proof of Theorem 3.1.3.

We assume from now on that $t_2 < \varepsilon^{-\alpha}$.

We have the following inequality :

$$\forall t \in [t_1, t_2], \quad \left(\frac{f(t)}{2}\right)^4 \geq K_4\varepsilon^2(1 + kt\ln(2+t)) \tag{3.4.18}$$

which implies that Lemma 3.4.4 can be applied with $r = f(t)/2$ for $t \in [t_1, t_2]$. Indeed, relation (3.4.18) is true since

$$K_4\varepsilon^2(1 + kt\ln(2+t)) \le K_4\varepsilon^2(1 + k\varepsilon^{-\alpha}\ln(2+\varepsilon^{-\alpha})) \le \left(\frac{\varepsilon^\beta}{4}\right)^4 \le \left(\frac{f(t)}{2}\right)^4$$

for ε small enough, as we chose $\alpha < 2 - 4\beta$. Lemma 3.4.4 yields that

$$\left(\varepsilon^{-\nu}\int_{|x-B|>f(t)/2}\omega(x,t)\mathrm{d}x\right)^{1/2} \le \left(C(k)\frac{\varepsilon^{k/2-\nu}}{(f(t)/2)^k}\right)^{1/2} \qquad \forall t \in [t_1, t_2]. \tag{3.4.19}$$

Since we chose k such that $k(1/2 - \beta) + 6\beta - 4 - \nu > 0$, for ε small enough we have that

$$\varepsilon^{-(k-6)\beta}\varepsilon^{k/2-4-\nu} = \varepsilon^{k(1/2-\beta)+6\beta-4-\nu} \le \frac{1}{C(k)2^{2k-6}}.$$

Recalling that $f(t) \ge \varepsilon^\beta/2$ for every $t \in [t_1, t_2]$, we infer that

$$f^{k-6}(t) \ge \left(\frac{\varepsilon^\beta}{2}\right)^{k-6} \ge C(k)2^k\varepsilon^{k/2-4-\nu}$$

which in turns gives that

$$C(k)\frac{\varepsilon^{k/2-\nu}}{(f(t)/2)^k} \le \frac{\varepsilon^4}{f^6(t)}. \tag{3.4.20}$$

Using relations (3.4.19) and (3.4.20) in (3.4.16) yields that

$$f'(t) \le C\varepsilon^\beta f(t) + C\frac{\varepsilon^2}{f^3(t)}.$$

which implies that

$$(f^4)'(t) \le C_1\varepsilon^\beta f^4(t) + C_1\varepsilon^2$$

for some constant C_1. The end of the argument is now straightforward. We use the Gronwall lemma to obtain that

$$f^4(t_2) \le f^4(t_1)e^{C_1\varepsilon^\beta(t_2-t_1)} + \varepsilon^{2-\beta}(e^{C_1\varepsilon^\beta(t_2-t_1)} - 1).$$

Since $t_2 - t_1 \le \varepsilon^{-\alpha}$ we have that $C_1\varepsilon^\beta(t_2 - t_1) \le C_1\varepsilon^{\beta-\alpha} < 1$ for ε small enough. Using the inequality $e^x \le 1 + 2x$ for $0 \le x \le 1$ and recalling that $f(t_1) = \varepsilon^\beta/2$ and $f(t_2) = \varepsilon^\beta$ we have that

$$\varepsilon^{4\beta} \le (\varepsilon^\beta/2)^4 e^{C_1\varepsilon^{\beta-\alpha}} + 2C_1\varepsilon^{2-\alpha}.$$

which implies that

$$1 \le \frac{e^{C_1\varepsilon^{\beta-\alpha}}}{16} + 2C_1\varepsilon^{2-4\beta-\alpha}.$$

Since $\alpha < \min(\beta, 2 - 4\beta)$, the right hand side of this inequality goes to $1/16$ as $\varepsilon \to 0$. So we obtain a contradiction if ε is small enough. We thus proved that $t_2 \ge \varepsilon^{-\alpha}$ if ε is small enough.

In conclusion, for any $\beta < 1/2$, and for any $\alpha < \min(\beta, 2 - 4\beta)$, there exists ε_0 small enough such that for every $\varepsilon \in (0, \varepsilon_0)$, we have that $\tau_{\varepsilon,\beta} > \varepsilon^{-\alpha}$. This completes the proof of Theorem 3.1.3.

3.5 Final remarks

A natural question is the signification of the hypothesis $T'''(x_0) = 0$. As we already discussed, the condition $T''(x_0) = 0$ means that x_0 is a critical point of the map $\widetilde{\gamma}$. Such critical points always exist in a bounded simply connected domain Ω. But to have a strong result of confinement as we proved, we need more than just a critical point of $\widetilde{\gamma}$. Indeed, we observed in Section 3.3 that we should not expect a confinement time better than $|\ln \varepsilon|$ around unstable points. So we need the stationary point x_0 to be at least stable. But the hypothesis $T'''(x_0) = 0$ is stronger than the stability. Indeed, the stability is characterized by the condition $|T'''(x_0)| < 2|T'(x_0)|^3$ which is significantly weaker than $T'''(x_0) = 0$.

So the hypothesis $T'''(x_0) = 0$ is more than just stability. Because we obtained the explicit value of $D^2\widetilde{\gamma}(x_0)$, see relation (3.3.2), we see that this condition is equivalent to the fact $D^2\widetilde{\gamma}(x_0)$ is a multiple of the identity. This means that the orbit of a single point vortex in the neighborhood of x_0 is almost a circle. We don't know whether this condition is indeed necessary to have a strong confinement as proved in Theorem 3.1.3. In our proof we use some crucial cancellations to prove the estimate of the moment of inertia from Lemma 3.4.3 that we can't reproduce without the hypothesis $T'''(x_0) = 0$.

One can wonder about the existence of domains satisfying the condition $T'''(x_0) = 0$ in a stationary point x_0. We call these domains valid domains. Let us notice that if $T(x_0) = 0$ and $T''(x_0) = 0$, then the condition $T'''(x_0) = 0$ is equivalent to $(T^{-1})'''(0) = 0$. This means that the image $\Omega = f(D(0,1))$ by any biholomorphic map of the form $f(z) = x_0 + a_1 z + \sum_{k=4}^{\infty} a_k z^k$ of the unit disk is a valid domain : there exists a biholomorphic map $T : \Omega \to D(0,1)$ such that $T(x_0) = T''(x_0) = T'''(x_0) = 0$ (one can choose $T = f^{-1}$). However, checking that a map f of the form given above is indeed biholomorphic may not be an easy task.

Let us observe now that regular convex polygons are valid domains. Indeed, by the Schwarz-Christoffel formula (see [1]), there exists a conformal map f from the unit disk to the n sided regular polygon with vertices at $\omega_n = e^{i2\pi/n}$, and its derivative has the form

$$f'(z) = c \prod_{k=1}^{n} \left(1 - z\omega_n^{-k}\right)^{1-\frac{2}{n}}.$$

We can therefore obtain an explicit value for the second and third derivatives, and check that they vanish at 0. So regular convex polygons are valid domains.

On the other hand, an ellipse which is not a circle is not a valid domain. Indeed, we know from [52] that a conformal mapping from the disk to an ellipse of foci ± 1 mapping 0 to 0 has a Taylor expansion $f(z) = z + A_3 z^3 + \dots$ near 0 with $A_3 > 0$.

Another method to obtain valid domains is to analyze the effect of rotational invariance. Assume that the domain Ω is invariant by rotation of angle $\theta \in (0, 2\pi)$ around the point $x_0 = 0$. This means that $\widetilde{\gamma}_\Omega(x) = \widetilde{\gamma}_\Omega(e^{i\theta}x)$ for every $x \in \Omega$. Differentiating this relation and using relation (3.2.4) implies that

$$\nabla\widetilde{\gamma}_\Omega(e^{i\theta}x) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \nabla\widetilde{\gamma}_\Omega(x)$$

and thus $\nabla\widetilde{\gamma}_\Omega(0) = 0$ implying that 0 is a stationary point. Similarly, differentiating again yields that

$$\begin{aligned}
\partial_1^2\widetilde{\gamma}_\Omega(x) &= \partial_1^2[\widetilde{\gamma}_\Omega(e^{i\theta x})] = \cos^2\theta\,\partial_1^2\widetilde{\gamma}_\Omega(e^{i\theta}x) + \sin^2\theta\,\partial_2^2\widetilde{\gamma}_\Omega(e^{i\theta}x) + 2\cos\theta\sin\theta\,\partial_1\partial_2\widetilde{\gamma}_\Omega(e^{i\theta}x) \\
\partial_2^2\widetilde{\gamma}_\Omega(x) &= \partial_2^2[\widetilde{\gamma}_\Omega(e^{i\theta}x)] = \sin^2\theta\,\partial_1^2\widetilde{\gamma}_\Omega(e^{i\theta}x) + \cos^2\theta\,\partial_2^2\widetilde{\gamma}_\Omega(e^{i\theta}x) - 2\cos\theta\sin\theta\,\partial_1\partial_2\widetilde{\gamma}_\Omega(e^{i\theta}x) \\
\partial_1\partial_2\widetilde{\gamma}_\Omega(x) &= \partial_1\partial_2[\widetilde{\gamma}_\Omega(e^{i\theta}x)] = (\cos^2\theta - \sin^2\theta)\partial_1\partial_2\widetilde{\gamma}_\Omega(e^{i\theta}x) \\
&\qquad + \cos\theta\sin\theta(\partial_2^2\widetilde{\gamma}_\Omega(e^{i\theta}x) - \partial_1^2\widetilde{\gamma}_\Omega(e^{i\theta}x)).
\end{aligned}$$

We set $x = 0$ and we subtract the first equation above from the second equation. We get

$$(1 - \cos^2\theta + \sin^2\theta)(\partial_2^2\widetilde{\gamma}_\Omega(0) - \partial_1^2\widetilde{\gamma}_\Omega(0)) = -4\cos\theta\sin\theta\partial_1\partial_2\widetilde{\gamma}_\Omega(0).$$

Using this in the third equation yields after some calculations that

$$\theta = \pi \qquad \text{or} \qquad \partial_1\partial_2\widetilde{\gamma}_\Omega(0) = 0.$$

If $\partial_1\partial_2\widetilde{\gamma}_\Omega(0) = 0$ and $\theta \neq \pi$ we observe that $\partial_1^2\widetilde{\gamma}_\Omega(0) = \partial_2^2\widetilde{\gamma}_\Omega(0)$. Therefore we have that

$$\theta = \pi \qquad \text{or} \qquad \exists\lambda,\ D^2\widetilde{\gamma}_\Omega(0) = \lambda I_2.$$

From (3.3.2) we know that

$$D^2\widetilde{\gamma}_\Omega(0) = \frac{\mu^2}{\pi}I_2 + \frac{1}{2\pi}\begin{pmatrix} p & q \\ q & -p \end{pmatrix}.$$

Clearly $D^2\widetilde{\gamma}_\Omega(0)$ is a multiple of the identity if an only if $p = q = 0$. Therefore we have that either $\theta = \pi$ or $p = q = 0$. From the definition of p and q given after relation (3.3.2) we see that for any conformal map T mapping 0 to 0 the condition $p = q = 0$ is equivalent to $T'''(0) = 0$. So if $\theta \neq \pi$, then $T'''(0) = 0$ and the domain is therefore valid. This is another proof of the fact that regular polygons are valid domains, and it also gives us many other valid domains.

We used Mathematica to plot the boundary of some valid domains. We consider different functions $f(z) = a_1 z + \sum_{k=4}^N a_k z^k$, with $N \geq 4$ and $|a_1| > \sum_{k=4}^N k|a_k|$ in order to ensure that f is injective on the unit disk. We obtain a large class of valid domains, with the unit disk of course and small variations of it (figure 3.2), but also larger perturbation of the disk (figure 3.3), and even some very erratic domains (figure 3.4). Notice that these domains don't necessarily have symmetry properties. Finally, by using the rotational invariance property, we can plot more complicated boundaries without knowing the biholomorphic map, like in figure 3.5. We plotted images of the interval $[0, 1]$ by maps of the form $b(x) = r(x)e^{i2\pi(x+\theta(x))}$, with $r(x) > 0$. We choose θ and r to be $1/p$-periodic functions with $p > 2$ an integer. This way, b plots a closed curve in $\mathbb{C}$ that is invariant by rotation of angle $2\pi/p \in (0, \pi)$. If this curve does not self intersect and is smooth, then its interior is a valid domain.

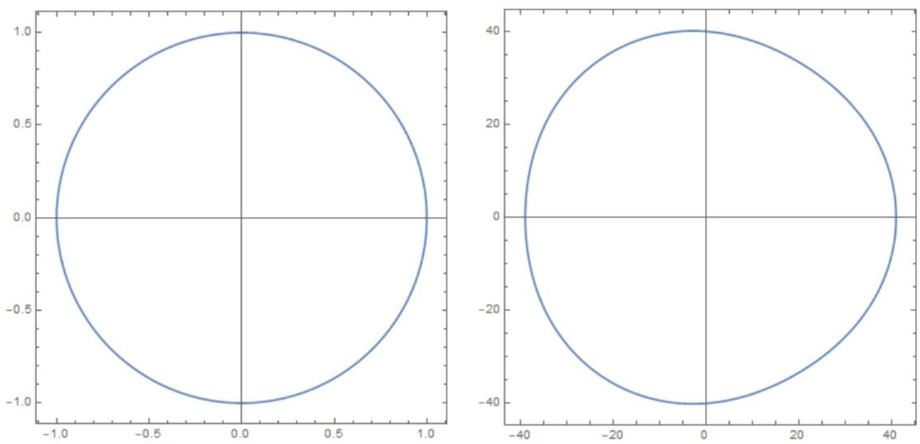


Figure 3.2 – Plot for $f(z) = z$ (left), and $f(z) = 40z + z^4$ (right).

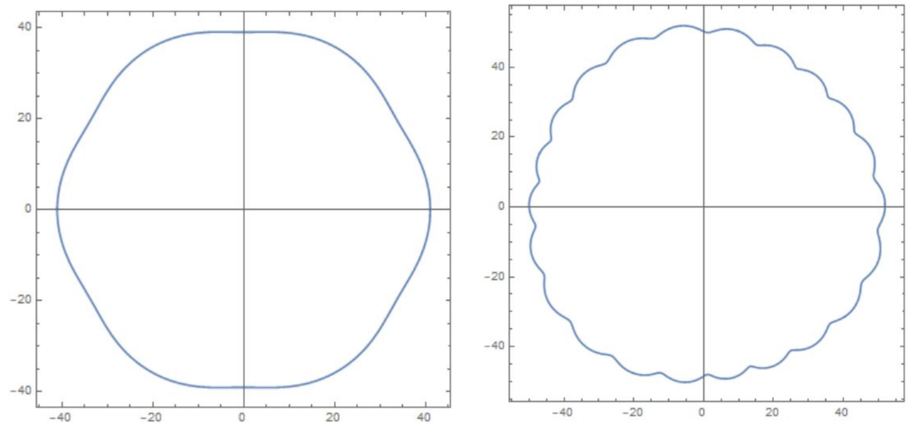


FIGURE 3.3 – Plot for $f(z) = 40z + z^7$ (left), and $f(z) = 50z + (i+1)z^4 + z^{23}$ (right).

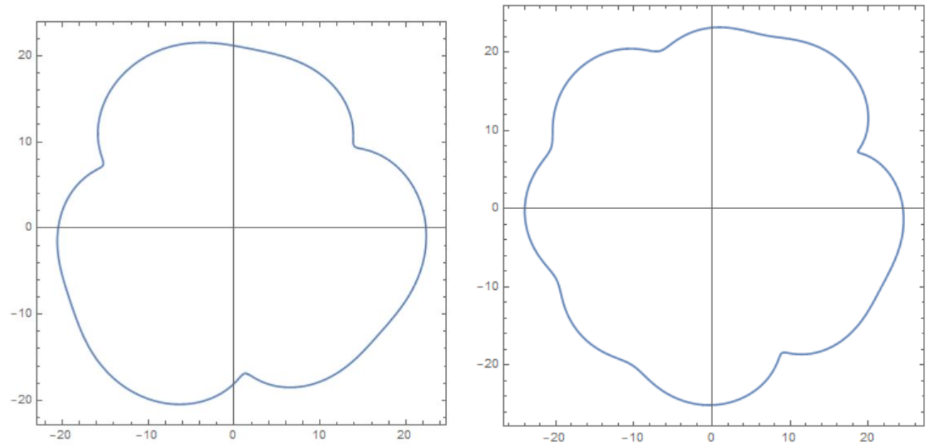


FIGURE 3.4 – Plot for $f(z) = 20z + (2i+1)z^4 + z^7$ (left), and $f(z) = 23z + iz^5 + iz^6 + z^9$ (right).

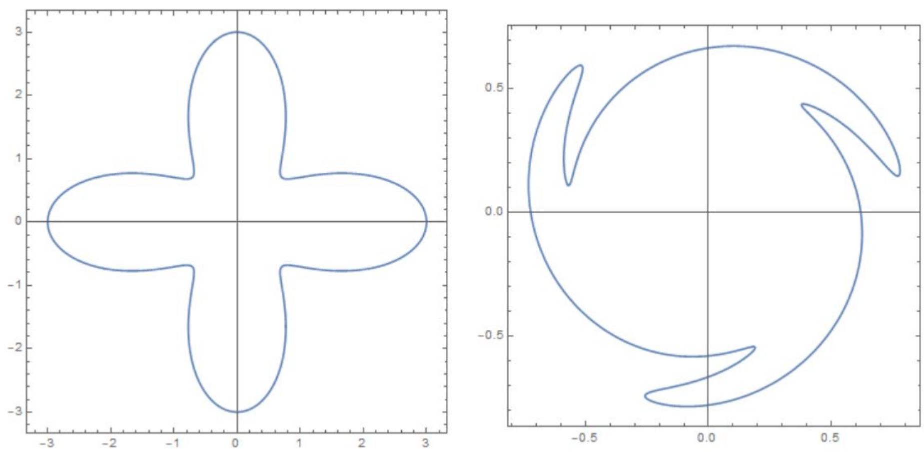


FIGURE 3.5 – Plot for $b(x) = (2+\cos(8\pi x))e^{i2\pi x}$ (left), and $b(x) = \dfrac{\exp\left(2i\pi(x+\frac{1}{8}\cos(6\pi x)\right)}{\frac{3}{2}+\frac{1}{4}\cos(6\pi x)}$ (right), with $x \in [0,1]$.

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Chapitre 4

Improbability of the collisions

This chapter is constituted of the following published article :

Two-dimensional point vortex dynamics in bounded domains : global existence for almost every initial data
SIAM J. Math. Anal., 2021, 54 (1), pp. 79–113.

In this paper, we prove that in bounded planar domains with $C^{2,\alpha}$ boundary, for almost every initial condition in the sense of the Lebesgue measure, the point vortex system has a global solution, meaning that there is no collision between two point-vortices or with the boundary. This extends the work previously done in [61] for the unit disk. The proof requires the construction of a regularized dynamics that approximates the real dynamics and some strong inequalities for the Green's function of the domain. In this paper, we make extensive use of the estimates given in [40]. We establish our relevant inequalities first in simply connected domains using conformal maps, then in multiply connected domains.

4.1 Introduction

Let us begin by recalling the Euler equations for two dimensional incompressible and inviscid fluids. Let Ω be an open, bounded and connected subset of $\mathbb{R}^2$. We denote by

$$u : \begin{cases} \Omega \times \mathbb{R}_+ & \to \mathbb{R}^2 \\ (x,t) & \mapsto u(x,t), \end{cases}$$

the velocity of a perfect incompressible fluid filling Ω. Then u must verify the incompressible Euler equations :

$$\begin{cases} \partial_t u(x,t) + u(x,t)\cdot\nabla u(x,t) = -\nabla p(x,t), & \forall (x,t)\in\Omega\times\mathbb{R}_+^* \\ u(x,0) = u_0(x), & \forall x\in\Omega \\ \nabla\cdot u(x,t) = 0, & \forall (x,t)\in\Omega\times\mathbb{R}_+ \\ u(x,t)\cdot n_\Omega(x) = 0, & \forall (x,t)\in\partial\Omega\times\mathbb{R}_+, \end{cases}$$

where p is the pressure within the fluid, n_Ω is the exterior normal unit vector to $\partial\Omega$ and u_0 is the initial velocity at the time $t=0$. Introducing the vorticity $\omega = \operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1$, the first equation of Euler's system gives the following equation for the vorticity :

$$\partial_t \omega(x,t) + u(x,t)\cdot\nabla\omega(x,t) = 0. \tag{4.1.1}$$

We can also express the velocity in terms of the vorticity thanks to the Biot-Savart law. When Ω is simply connected, the Biot-Savart law reads

$$u(x,t) = \int_\Omega \nabla_x^\perp G_\Omega(x,y)\omega(y,t)\mathrm{d}y, \tag{4.1.2}$$

where G_Ω is the Green's function of the domain Ω. It is important to recall that the Green's function of any open and connected subset of $\mathbb{R}^2$ - we call them *domains* in this paper - can be decomposed as

$$G_\Omega = G_{\mathbb{R}^2} + \gamma_\Omega, \tag{4.1.3}$$

where γ_Ω is a smooth function in $\Omega \times \Omega$. Indeed, $\gamma_\Omega(x,y)$ is harmonic in both variables.

We define the point vortex system as in [62]. We assume that at the initial time, the vorticity is a sum of Dirac masses $\omega_0 = \sum_{i=1}^N a_i \delta_{x_i}$ where N is an integer greater than 1, which we fix for the rest of this paper, and the masses a_i are real numbers, also fixed. Since the vorticity equation (4.1.1) is a transport equation in ω, we expect the vorticity to remain a sum of Dirac masses with the same intensity as at the initial time. So we choose to write $\omega(t) = \sum_{i=1}^N a_i \delta_{x_i(t)}$. We then define the point vortex system in a simply connected domain as the solution of the system of equations :

$$\forall 1 \leq i \leq N, \ \ \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = \sum_{\substack{j=1 \\ j \neq i}}^N \nabla_x^\perp G_\Omega(x_i(t), x_j(t)) a_j + \nabla_x^\perp \gamma_\Omega(x_i(t), x_i(t)) a_i. \tag{4.1.4}$$

This is obtained by introducing the expression of the vorticity and the decomposition (4.1.3) into the Biot-Savart law (4.1.2) and by removing the singular term that appears in the limit of $\nabla_x G_{\mathbb{R}^2}(x, x_i(t)) = \frac{(x-x_i(t))^\perp}{2\pi|x-x_i(t)|^2}$ when x goes to $x_i(t)$. This term represents high speed rotation around $x_i(t)$, so it shouldn't affect the motion of $x_i(t)$ itself.

All those choices have been mathematically justified in [63], where it has been proved that highly concentrated smooth solution of the Euler equations converges in the sense of measures to the solution of the point vortex system, as the initial data converges towards the initial sum of Dirac masses. It has also been proved in [36] that the point vortex system is a good approximation of the Euler equation from the point of view of numerics, taking as initial data a grid of vortices approaching a smooth initial vorticity.

The question that naturally arises now is whether the system of equations defined in (4.1.4) has a global solution for every initial condition $(x_i(0))_i$. The answer to that question unfortunately is negative in general, since in $\mathbb{R}^2$ one can build an initial datum such that point vortices collapse in finite time. See [62], [46], or [50] for explicit examples. By construction, the point vortex dynamics isn't defined anymore as soon as a collapse occurs, since equation (4.1.4) becomes singular when two points collide. But what we can expect is that these occurrences of collapse are exceptional, meaning that the initial configurations leading to collapse are negligible in the sense of the Lebesgue measure. This result has been proved in [61] in the unit disk $D(0,1)$. In the case of $\mathbb{R}^2$, it has been proved with the additional assumption that every possible sum of the masses never vanishes, meaning that $\sum_{i\in P} a_i \neq 0$ for every $P \subset \{1,\ldots,n\}$. Proofs of these results can be found in [61] and [62]. Very recently, [34] proved that the assumption that $\sum_{i=1}^N a_i \neq 0$ in $\mathbb{R}^2$ can be removed.

Let us give a precise statement of the result of [61] in the case of the disk.

Théorème 4.1.1. *If $\Omega = D(0,1)$ the open unit disk, then the point vortex dynamics* (4.1.4) *for any fixed number of points $N \geq 1$ and masses $(a_i)_i \in \mathbb{R}^N$ is globally well defined except maybe for a set of initial conditions in Ω^N which has vanishing Lebesgue measure.*

The purpose of this article is to prove a generalization of this theorem to more general bounded domains. Let Ω be an open bounded and connected subset of $\mathbb{R}^2$ with a $C^{2,\alpha}$ boundary for some $\alpha > 0$. In the case where Ω is multiply connected, the point vortex dynamics is changed since the Biot Savart law is different. We refer to [27, Chapter 15] for the point-vortex system in multiply-connected domains. We will give all the details in section 4.3 but the result is that for a domain Ω that has m holes, the point vortex dynamics is given by

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = \sum_{\substack{j=1\\ j\neq i}}^{N} \nabla_x^\perp G_\Omega(x_i(t), x_j(t))a_j + \nabla_x^\perp \gamma_\Omega(x_i(t), x_i(t))a_i + \sum_{j=1}^{m} c_j(t)\beta_j(x_i) \tag{4.1.5}$$

for all $1 \leq i \leq N$. Above

$$c_j(t) = \xi_j + \sum_{k=1}^{N} a_k w_j(x_k(t)),$$

ξ_j is the circulation of the velocity u on the boundary of the j-th hole of Ω, w_j are the harmonic measures and β_j are the basis of the harmonic vector fields of the domain Ω of circulation $\delta_{j,\ell}$ on Γ_ℓ, for $1 \leq \ell \leq m$.

Let us observe that by the Kelvin theorem, the circulations ξ_j are constant in time. They are therefore prescribed at the initial time.

Let λ be the Lebesgue measure on $\mathbb{R}^{2N}$. We define on the set $\overline{\Omega}^N$:

$$d(X) = \min\left(\min_{i\neq j} |x_i - x_j|, \min_i \mathrm{d}(x_i, \partial\Omega)\right) \quad \forall X = (x_1, \ldots, x_N).$$

We define $\Gamma = \{X = (x_1, \ldots, x_N), d(X) > 0\}$. This is the set of all configurations for which relation (4.1.5) makes sense. We note by $S_t X$ the evolved configuration by the dynamics (4.1.5) from the starting configuration $X \in \Gamma$, after a time t. We know there exists a time $\tau(X) = \sup\{t \geq 0, S_t X \in \Gamma\} > 0$ until which the dynamics is well defined. In this paper we will prove the following theorem.

Théorème 4.1.2. *Let Ω be an open, bounded and connected subset of $\mathbb{R}^2$ with a $C^{2,\alpha}$ boundary for some $\alpha > 0$ with $m \in \mathbb{N}$ holes. We fix the number of point vortices $N \geq 1$, the masses $(a_i)_{1\leq i\leq N} \in \mathbb{R}^N$, and the circulations $(\xi_j)_{1\leq j\leq m} \in \mathbb{R}^m$. With the previous notations we have that*

$$\lambda(\{X \in \Omega^N, \tau(X) < \infty\}) = 0,$$

meaning that for almost every starting position X, the point vortex dynamics in Ω is well defined for every time.

We observe that Theorem 4.1.2 has a simple proof in the case of a single point vortex in a simply connected domains, that is in the case $N = 1$ and $m = 0$. We introduce the Robin function $\widetilde{\gamma}_\Omega(x) = \gamma_\Omega(x, x)$. Since $\gamma_\Omega(x, y) = \gamma_\Omega(y, x)$, we have that $\nabla\widetilde{\gamma}_\Omega(x) = \nabla_x \gamma_\Omega(x, x) + \nabla_y \gamma_\Omega(x, x) = 2\nabla_x \gamma_\Omega(x, x)$. In this case the dynamics of a single point vortex becomes

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{1}{2}\nabla^\perp \widetilde{\gamma}_\Omega(x(t))a.$$

Therefore, a single point vortex evolves on the level set of the Robin function. This map has been studied in [40], from which we know in particular that $\widetilde{\gamma}_\Omega(x) \underset{x\to\partial\Omega}{\longrightarrow} +\infty$. Therefore a single point vortex can't hit the boundary so the dynamics is well defined for every time.

The proof of Theorem 4.1.2 that we will give in section 4.4.2 borrows arguments from [61], but we have to deal with two major difficulties. The first one is the construction of a convenient regularized dynamics, and the second one is to prove some analytic inequalities on the Green's function and the Robin function of the domain Ω. In Section 4.2 we will obtain the required inequalities for those maps in the case of simply connected domains, and in Section 4.3, we will extend those results to the case of multiply connected domains. Section 4.4 is devoted to the construction of the regularized dynamics and the completion of the proof of Theorem 4.1.2.

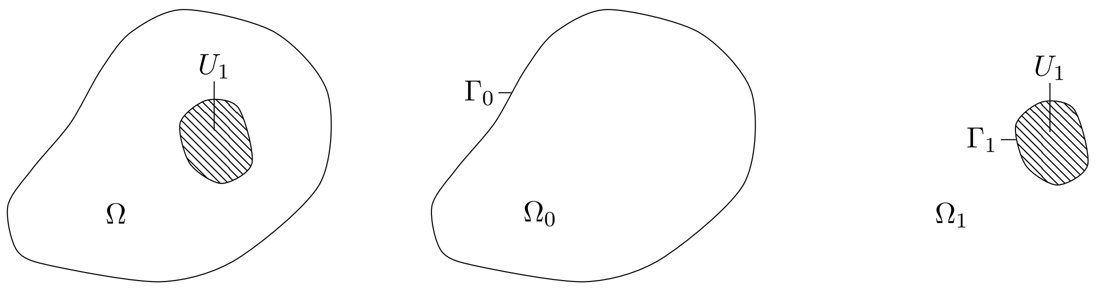

FIGURE 4.1 – An example for $m = 1$. The domains Ω, Ω_0 and Ω_1 and their boundaries.

4.2 Simply connected and exterior domains

List of notations :
— $N \in \mathbb{N}$ denotes the number of point vortices ;
— λ is the Lebesgue measure on $\mathbb{R}^{2N}$;
— $(x_1, x_2)^\perp = (-x_2, x_1)$;
— Ω is a $C^{2,\alpha}$ bounded domain of $\mathbb{R}^2$ with $m \in \mathbb{N}$ holes, and its boundaries are Γ_j, $0 \le j \le m$, with Γ_0 the exterior boundary ;
— $\mathcal{U}$ denotes a general bounded domain with $C^{2,\alpha}$ boundary ;
— U denotes a general simply connected bounded domain with $C^{2,\alpha}$ boundary ;
— Π denotes a general exterior domain with $C^{2,\alpha}$ boundary ;
— $D(x_0, r)$ is the disk of center x_0 and of radius r and $D = D(0, 1)$;
— $\Pi_D = \left(\overline{D(0,1)}\right)^c$;
— T denotes a biholomorphic map, usually from U to D or from Π to Π_D ;
— $n_{\mathcal{U}}$ is the exterior normal unit vector to $\partial\mathcal{U}$, extended to a neighborhood of $\partial\mathcal{U}$ by relation (4.2.17) when possible ;
— $G_{\mathcal{U}}$ is the Green's function of the domain $\mathcal{U}$, and $G = G_\Omega$ in section 4.4 ;
— $\gamma_{\mathcal{U}}$ is the regular part of $G_{\mathcal{U}}$, see relation (4.1.3), and $\gamma = \gamma_\Omega$ in section 4.4 ;
— $\widetilde{\gamma}_{\mathcal{U}}(x) = \gamma_{\mathcal{U}}(x, x)$ is the Robin function of the domain $\mathcal{U}$, and $\widetilde{\gamma} = \widetilde{\gamma}_\Omega$ in section 4.4 ;
— $C, C_1, C_2, \ldots$, are strictly positive constants that may vary from one line to another, when their value is not important to the result ;
— $a \cdot b$ is the scalar product of vectors in $\mathbb{R}^2$;
— ∇f and $\nabla \cdot g$ are respectively the gradient of f and the divergence of g ;
— V_j are neighborhoods of Γ_j, and K is a compact set as in the decomposition (4.3.1) ;
— $S_t X$ is the solution of the point vortex dynamics starting from X after a time t, and $S_t^\varepsilon X$ the regularized dynamics constructed in Section 4.4.1 ;

For the rest of this paper, Ω denotes a bounded domain whose boundary is $C^{2,\alpha}$ for some $0 < \alpha < 1$. It is either simply connected or it has $m \in \mathbb{N}$ holes that are the simply connected bounded domains $U_1, \ldots, U_m$ and their boundaries are $\Gamma_1, \ldots, \Gamma_m$. We denote by Γ_0 the exterior boundary of Ω, meaning that Ω lies within the interior in the Jordan sense of Γ_0, and by Ω_0 the simply connected bounded domain whose boundary is Γ_0, namely the domain Ω "without holes". Finally, we call *exterior domain* a domain whose complement is bounded and simply connected. We denote for $1 \le j \le m$, $\Omega_j = (\overline{U_j})^c$ the exterior domains of the m holes. The domain Ω is pictured in Figure 4.1.

Since $\Omega = \bigcap_{j=0}^m \Omega_j$, with Ω_0 simply connected, and where Ω_j, $j \ge 1$ are exterior domains, our strategy in this paper is to establish inequalities on Ω by establishing them for any bounded and simply connected domain U, and for any exterior domain Π.

We also denote by $D = D(0, 1)$ the unit disk, and by Π_D the exterior domain of D, namely $\Pi_D = \{x \in \mathbb{C}, |x| > 1\}$.

4.2.1 Holomorphic maps

Holomorphic maps are the subject of the Chapter 2 of [1]. A map $T : \mathbb{C} \to \mathbb{C}$ is a *biholomorphic map* if T and T^{-1} are holomorphic maps. Such maps satisfy that their derivative never vanishes.

Let us write $T = T_1 + iT_2$, and we identify $\mathbb{R}^2$ and $\mathbb{C}$, meaning that we also denote $T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$. Then a holomorphic map satisfies the Cauchy-Riemann equations

$$\begin{cases} \partial_1 T_1 = \partial_2 T_2 \\ \partial_1 T_2 = -\partial_2 T_1. \end{cases}$$

We have that

$$T' = \partial_1 T = -i\partial_2 T$$

and therefore

$$T'' = \partial_1^2 T = -\partial_2^2 T = -i\partial_1\partial_2 T.$$

Finally, the Jacobian matrix of T is $JT = \begin{pmatrix} \partial_1 T_1 & \partial_2 T_1 \\ \partial_1 T_2 & \partial_2 T_2 \end{pmatrix}$, so $\det JT = |T'|^2$. In the following, we will freely use these properties. In particular, we can always substitute the second partial derivative ∂_2 with $i\partial_1$ according to these formulas. For any map $f \in C^1(\mathbb{C}, \mathbb{R})$, we have that

$$\nabla(f \circ T)(x) = \begin{pmatrix} \partial_1 T_1(x)\partial_1 f(T(x)) + \partial_1 T_2(x)\partial_2 f(T(x)) \\ -\partial_1 T_2(x)\partial_1 f(T(x)) + \partial_1 T_1(x)\partial_2 f(T(x)) \end{pmatrix}. \tag{4.2.1}$$

We conclude this paragraph with a technical lemma.

Lemme 4.2.1. *For any bounded domain $\mathcal{U}$ whose boundary is $C^{2,\alpha}$, and for any $\kappa < 1$, we have that*

$$\iint_{\mathcal{U}\times\mathcal{U}} \frac{1}{|x-y|^{1+\kappa}} \mathrm{d}x\mathrm{d}y < \infty,$$

and

$$\iint_{\mathcal{U}\times\mathcal{U}} \frac{1}{d(x,\partial\mathcal{U})^{\kappa}} \frac{1}{|x-y|} \mathrm{d}x\mathrm{d}y < \infty.$$

Moreover there exists a constant C depending only on $\mathcal{U}$ such that for sufficiently small $\varepsilon > 0$,

$$\int_{\{x\in\mathcal{U}, d(x,\partial\mathcal{U})\geq\varepsilon\}} \frac{1}{d(x,\partial\mathcal{U})^{1+\kappa}} \mathrm{d}x \leq C\varepsilon^{-\kappa}. \tag{4.2.2}$$

Démonstration. Let $R = \operatorname{diam}(\mathcal{U})$ so that $\mathcal{U} \subset B(x,R)$ for every $x \in \mathcal{U}$. Then

$$\iint_{\mathcal{U}\times\mathcal{U}} \frac{1}{|x-y|^{1+\kappa}} \mathrm{d}x\mathrm{d}y \leq \int_{\mathcal{U}} \int_{B(x,R)} \frac{1}{|x-y|^{1+\kappa}} \mathrm{d}x\mathrm{d}y = |\mathcal{U}| \int_0^R \int_0^{2\pi} \frac{1}{r^{1+\kappa}} r\mathrm{d}\theta\mathrm{d}r < \infty.$$

With the same argument,

$$\iint_{\mathcal{U}\times\mathcal{U}} \frac{1}{d(x,\partial\mathcal{U})^{\kappa}} \frac{1}{|x-y|} \mathrm{d}x\mathrm{d}y \leq 2\pi R \int_{\mathcal{U}} \frac{1}{d(x,\partial\mathcal{U})^{\kappa}} \mathrm{d}x.$$

To prove that the integral $\int_{\mathcal{U}} \frac{1}{d(x,\partial\mathcal{U})^{\kappa}} \mathrm{d}x$ is finite, and to prove (4.2.2), we make a finite number of local changes of coordinates and we use that $\mathcal{U}$ is bounded to write

$$\int_{\mathcal{U}} \frac{1}{d(x,\partial\mathcal{U})^{\kappa}} \mathrm{d}x \leq C \int_0^R \frac{1}{s^{\kappa}} \mathrm{d}s < \infty$$

and

$$\int_{\{x\in\mathcal{U}, d(x,\partial\mathcal{U})\geq\varepsilon\}} \frac{1}{d(x,\partial\mathcal{U})^{1+\kappa}} \mathrm{d}x \leq \int_\varepsilon^R \frac{1}{s^{1+\kappa}} \mathrm{d}s \leq C\varepsilon^{-\kappa}.$$

□

4.2.2 The Riemann Mapping Theorem

We refer now to Chapter 6 of [1].

Théorème 4.2.2 (Riemann Mapping Theorem)**.** *For any non empty, open and simply connected subset U of $\mathbb{C}$, that isn't the whole plane, there exists a biholomorphism from U to the unit disk D.*

The consequence of this theorem is that any suitable domain is linked to the disk by a map that has very interesting properties related to the Green's function of both domains. However the Riemann Mapping Theorem only states the theoretical existence of such map, and only a few explicit examples are known. In particular, we have no control over the derivatives of the biholomorphism in general. We combine Theorems 3.5 and 3.6 from [71] to obtain the following corollary of the Kellogg-Warschawski Theorem.

Théorème 4.2.3. *Let T be a biholomorphism mapping on a bounded, open, and simply connected set U whose boundary ∂U is a $C^{2,\alpha}$ Jordan curve, with $0 < \alpha < 1$. Then T, T' and T'' are continuous up to $\overline{U}$, and T^{-1} and $(T^{-1})'$ are continuous up to $\overline{T(U)}$. Thus there exist constants m and M satisfying for every $x \in \overline{U}$ that $0 < m \leq |T'(x)| \leq M$ and $|T''(x)| \leq M$.*

Let us stress the fact that since the automorphisms of the disk are known explicitly and belong to $C^\infty(\overline{D})$, if there exists one biholomorphism $T : U \to D$ that is smooth up to the boundary, then every biholomorphism $T : U \to D$ is smooth up to the boundary. Since in this paper we will always consider smooth domains U, every biholomorphism $T : U \mapsto D$ will satisfy the conclusions of Theorem 4.2.3.

Please note that the $C^{2,\alpha}$ condition is not optimal. For instance, it is known that if ∂U has a parametrization with a Dini-continuous curvature, then the conclusion of the theorem is still true. Also, assuming that $\partial U \in C^{n,\alpha}$ implies more generally that $T^{(n)}$ is continuous up to ∂U. Despite these remarks, we will stick to the condition $C^{2,\alpha}$ in the context of this article.

In conclusion, the Riemann Mapping Theorem states the existence of the map $T : U \to D$ and Theorem 4.2.3 states that $\forall x \in \overline{U}, 0 < m \leq |T'(x)| \leq M$ and $|T''(x)| \leq M$.

We have a very similar result, this time concerning *exterior domains*, which we define as the complement of the closure of a bounded and simply connected set in $\mathbb{C}$. We have the following theorem.

Théorème 4.2.4. *Let Π be an exterior domain, with $C^{2,\alpha}$ boundary. Let T be a biholomorphic map from Π to $\Pi_D = \{x \in \mathbb{C}, |x| > 1\}$. Such a map exists and satisfies that T, T' and T'' are continuous up to $\overline{\Pi}$, and that T^{-1} and $(T^{-1})'$ are continuous up to $\overline{T(\Pi)}$. Moreover there exist constants m and M such that $\forall x \in \Pi$, $0 < m \leq |T'(x)| \leq M$ and $|T''(x)| \leq M$.*

The proof of this result can be found in [47]. It follows from the bounded domain case using the holomorphic map $T : \Pi_D \to D$, $T(z) = \frac{1}{z}$.

4.2.3 Green's Function

We start by recalling that for every $(x, y) \in D \times D$, $x \neq y$,

$$G_D(x, y) = \frac{1}{2\pi} \ln \frac{|x - y|}{|x - y^*||y|}, \tag{4.2.3}$$

where $y^* = \frac{y}{|y|^2}$ is the inverse of y relative to the unit circle. Using the decomposition (4.1.3) we have that for every $(x, y) \in D \times D$, $x \neq y$,

$$\gamma_D(x, y) = G_D(x, y) - G_{\mathbb{R}^2}(x, y) = -\frac{1}{2\pi} \ln(|x - y^*||y|) \tag{4.2.4}$$

and by continuity the relation $\gamma_D(x,y) = -\frac{1}{2\pi}\ln(|x-y^*||y|)$ holds true also for $y = x$. For every $x \in D$ we thus have that

$$\widetilde{\gamma}_D(x) = \gamma_D(x,x) = -\frac{1}{2\pi}\ln||x|^2 - 1|. \tag{4.2.5}$$

Notice that $\widetilde{\gamma}_D$ is a radial function.

Let U be a bounded and simply connected domain. If $T : U \to D$ is a biholomorphic map, then we have the following property.

Proposition 4.2.5. *For every $(x,y) \in U \times U$, $x \neq y$,*

$$G_U(x,y) = G_D(T(x),T(y)) = \frac{1}{2\pi}\ln\frac{|T(x)-T(y)|}{|T(x)-T(y)^*||T(y)|}.$$

Using this in the decomposition (4.1.3) we obtain

$$\gamma_U(x,y) + G_{\mathbb{R}^2}(x,y) = \gamma_D(T(x),T(y)) + G_{\mathbb{R}^2}(T(x),T(y)) \tag{4.2.6}$$

and thus

$$\begin{aligned}\forall x \in U, \quad \widetilde{\gamma}_U(x) &= \lim_{y\to x}\gamma_U(x,y)\\ &= \lim_{y\to x}\left(\gamma_D(T(x),T(y)) + \frac{1}{2\pi}\ln\frac{|T(x)-T(y)|}{|x-y|}\right).\end{aligned}$$

Therefore

$$\forall x \in U, \quad \widetilde{\gamma}_U(x) = \widetilde{\gamma}_D(T(x)) + \frac{1}{2\pi}\ln|T'(x)|. \tag{4.2.7}$$

A quite remarkable fact is that for every $(x,y) \in \Pi_D, x \neq y$,

$$G_{\Pi_D}(x,y) = \frac{1}{2\pi}\ln\frac{|x-y|}{|x-y^*||y|}, \tag{4.2.8}$$

which is the same expression as for $G_D(x,y)$. Thus the relations above also hold true for any exterior domain Π and any biholomorphism $T : \Pi \to \Pi_D$, which exists according to Theorem 4.2.4. For example, for every $(x,y) \in \Pi \times \Pi$, $x \neq y$,

$$G_\Pi(x,y) = G_{\Pi_D}(T(x),T(y)) = \frac{1}{2\pi}\ln\frac{|T(x)-T(y)|}{|T(x)-T(y)^*||T(y)|}.$$

Let us recall here a classical theorem, see for example [9, Theorem 4.17].

Théorème 4.2.6. *Let $\mathcal{U}$ be a bounded domain with $C^{2,\alpha}$ boundary. Then $G_{\mathcal{U}} \in C^2\big(\overline{\mathcal{U}} \times \overline{\mathcal{U}} \setminus \{(x,x), x \in \overline{\mathcal{U}}\}\big)$.*

In other words, except where $x = y$, the Green's function is smooth up to the boundary.

In their proof of Theorem 4.1.1, the authors of [61] show that for any $\kappa < 1$,

$$\iint_{D\times D}\frac{1}{d(x,\partial D)^\kappa}|\nabla_x G_D(x,y)\cdot\nabla^\perp\widetilde{\gamma}_D(x)| < \infty \tag{4.2.9}$$

and

$$\iint_{D\times D}\frac{1}{|x-y|^\kappa}|\nabla_x G_D(x,y)\cdot\nabla^\perp\widetilde{\gamma}_D(x)| < \infty. \tag{4.2.10}$$

In this paper we will extend these inequalities to the more general bounded domain Ω.

4.2.4 Intermediate lemmas

We recall that U is a bounded simply connected domain, and T is a biholomorphism from U to D.

The following lemma is the first step to extend inequalities (4.2.9) and (4.2.10) to the domain U.

Lemme 4.2.7. *For every x and y in U :*

$$\nabla_x G_U(x,y)\cdot\nabla^\perp\widetilde{\gamma}_U(x) = |T'(x)|^2\nabla_x G_D(T(x),T(y))\cdot\nabla^\perp\widetilde{\gamma}_D(T(x)) + \frac{\nabla_x G_D(T(x),T(y))\cdot\psi(x)}{|T'(x)|^2}, \tag{4.2.11}$$

with $\psi : U \to \mathbb{R}^2$ an explicit bounded function on U.

Démonstration. From Proposition 4.2.5 and relation (4.2.1), we have that

$$\nabla_x G_U(x,y) = \begin{pmatrix} \partial_{x_1}G_D(T(x),T(y))\partial_1 T_1(x) + \partial_{x_2}G_D(T(x),T(y))\partial_1 T_2(x) \\ -\partial_{x_1}G_D(T(x),T(y))\partial_1 T_2(x) + \partial_{x_2}G_D(T(x),T(y))\partial_1 T_1(x) \end{pmatrix}.$$

Similarly from relation (4.2.7) and relation (4.2.1), since T' is also a holomorphic map, we have that

$$\nabla\widetilde{\gamma}_U(x) = \begin{pmatrix} \partial_1\widetilde{\gamma}_D\partial_1 T_1 + \partial_2\widetilde{\gamma}_D\partial_1 T_2 + \dfrac{\partial_1 T_1\partial_1^2 T_1 + \partial_1 T_2\partial_1^2 T_2}{2\pi|T'|^2} \\ -\partial_1\widetilde{\gamma}_D\partial_1 T_2 + \partial_2\widetilde{\gamma}_D\partial_1 T_1 + \dfrac{-\partial_1 T_1\partial_1^2 T_2 + \partial_1 T_2\partial_1^2 T_1}{2\pi|T'|^2} \end{pmatrix}. \tag{4.2.12}$$

Therefore,

$$\begin{aligned}\nabla_x G_U\cdot\nabla^\perp\widetilde{\gamma}_U = & -(\partial_{x_1}G_D\partial_1 T_1 + \partial_{x_2}G_D\partial_1 T_2) \\ & \times\left(-\partial_1\widetilde{\gamma}_D\partial_1 T_2 + \partial_2\widetilde{\gamma}_D\partial_1 T_1 + \frac{-\partial_1 T_1\partial_1^2 T_2 + \partial_1 T_2\partial_1^2 T_1}{2\pi|T'|^2}\right) \\ & + (-\partial_{x_1}G_D\partial_1 T_2 + \partial_{x_2}G_D\partial_1 T_1) \\ & \times\left(\partial_1\widetilde{\gamma}_D\partial_1 T_1 + \partial_2\widetilde{\gamma}_D\partial_1 T_2 + \frac{\partial_1 T_1\partial_1^2 T_1 + \partial_1 T_2\partial_1^2 T_2}{2\pi|T'|^2}\right).\end{aligned}$$

We notice that the terms with the factor $\partial_{x_1}G_D\partial_1\widetilde{\gamma}_D$ cancel each others, as well as the terms with $\partial_{x_2}G_D\partial_2\widetilde{\gamma}_D$. We thus have that

$$\begin{aligned}\nabla_x G_U\cdot\nabla^\perp\widetilde{\gamma}_U = & -\frac{1}{2\pi}(\partial_{x_1}G_D\partial_1 T_1 + \partial_{x_2}G_D\partial_1 T_2)\left(\frac{-\partial_1 T_1\partial_1^2 T_2 + \partial_1 T_2\partial_1^2 T_1}{|T'|^2}\right) \\ & + \frac{1}{2\pi}(-\partial_{x_1}G_D\partial_1 T_2 + \partial_{x_2}G_D\partial_1 T_1)\left(\frac{\partial_1 T_1\partial_1^2 T_1 + \partial_1 T_2\partial_1^2 T_2}{|T'|^2}\right) \\ & - \partial_{x_1}G_D\partial_2\widetilde{\gamma}_D(\partial_1 T_1)^2 + \partial_{x_2}G_D\partial_1\widetilde{\gamma}_D(\partial_1 T_2)^2 \\ & - \partial_{x_1}G_D\partial_2\widetilde{\gamma}_D(\partial_1 T_2)^2 + \partial_{x_2}G_D\partial_1\widetilde{\gamma}_D(\partial_1 T_1)^2.\end{aligned}$$

The last two rows can be simplified, showing that they are equal to $|T'|^2\nabla_x G_D\cdot\nabla^\perp\widetilde{\gamma}_D$. For the first two rows, we factor out by $\partial_{x_i}G_D$ and then by $\partial_1^2 T_i$, so that

$$\begin{aligned}\nabla_x G_U\cdot\nabla^\perp\widetilde{\gamma}_U = & \frac{\partial_{x_1}G_D}{2\pi|T'|^2}[-2\partial_1 T_1\partial_1 T_2\partial_1^2 T_1 + ((\partial_1 T_1)^2 - (\partial_1 T_2)^2)\partial_1^2 T_2] \\ & + \frac{\partial_{x_2}G_D}{2\pi|T'|^2}[(-(\partial_1 T_2)^2 + (\partial_1 T_1)^2)\partial_1^2 T_1 + 2\partial_1 T_1\partial_1 T_2\partial_1^2 T_2] \\ & + |T'|^2\nabla_x G_D\cdot\nabla^\perp\widetilde{\gamma}_D.\end{aligned}$$

This proves equality (4.2.11) with the following explicit function ψ :

$$\psi(x) = \frac{1}{2\pi}\begin{pmatrix}-2\partial_1 T_1\partial_1 T_2\partial_1^2 T_1 + ((\partial_1 T_1)^2 - (\partial_1 T_2)^2)\partial_1^2 T_2\\ (-(\partial_1 T_2)^2 + (\partial_1 T_1)^2)\partial_1^2 T_1 + 2\partial_1 T_1\partial_1 T_2\partial_1^2 T_2\end{pmatrix}. \tag{4.2.13}$$

The function ψ is bounded since, thanks to Theorem 4.2.3, both T' and T'' are bounded. □

Notice that the proof of this lemma only uses Proposition 4.2.5 and relation (4.2.7), so the lemma also holds in the domain Π. More precisely, if T denotes this time a biholomorphism from Π to Π_D we have that

$$\nabla_x G_\Pi(x,y)\cdot\nabla^\perp\widetilde{\gamma}_\Pi(x) = |T'(x)|^2\nabla_x G_{\Pi_D}(T(x),T(y))\cdot\nabla^\perp\widetilde{\gamma}_{\Pi_D}(T(x)) + \frac{\nabla_x G_{\Pi_D}(T(x),T(y))\cdot\psi(x)}{|T'(x)|^2},$$

where ψ is another bounded function on Π that has the same expression in terms of the conformal map T.

We specify now some properties of γ_U.

Lemme 4.2.8. *We have that for any $x_0\in\partial U$, $\gamma_U(x,y)\underset{x,y\to x_0}{\longrightarrow}+\infty$.*

Démonstration. Let $T: U\to D$ a biholomorphism. Relations (4.2.6) and (4.2.4) yield that

$$\gamma_U(x,y) = -\frac{1}{2\pi}\ln\big(|T(x)-T(y)^*||T(y)\big) + \frac{1}{2\pi}\ln\frac{|T(x)-T(y)|}{|x-y|}. \tag{4.2.14}$$

Obviously $|T(x)-T(y)^*||T(y)| \to |T(x_0)-T(x_0)^*||T(x_0)| = 0$ when x and y go to x_0 so $-\frac{1}{2\pi}\ln\big(|T(x)-T(y)^*||T(y)|\big)$ goes to $+\infty$. Therefore we only need to obtain a lower bound for the other term. By Theorem 4.2.3 the map $x,y\mapsto\frac{|T(x)-T(y)|}{|x-y|}$ is continuous and non zero on $\overline{U}\times\overline{U}$. Therefore, there exists a constant C such that

$$\frac{|T(x)-T(y)|}{|x-y|} > C, \tag{4.2.15}$$

for all $x,y\in\overline{U}$ and thus the lemma is proved. □

Noticing that for any exterior domain Π and biholomorphism $T:\Pi\to\Pi_D$ there exists a neighborhood of $\partial\Pi\times\partial\Pi$ in $\Pi\times\Pi$ and a constant C such that in this neighborhood relation (4.2.15) holds, the proof of the previous lemma holds for exterior domains too and thus,

$$\forall x_0\in\partial\Pi,\ \ \gamma_\Pi(x,y)\underset{x,y\to x_0}{\longrightarrow}+\infty. \tag{4.2.16}$$

The following lemma gives explicit estimates of $d(x,\partial U)$, $d(y,\partial U)$ and $|x-y|$ when $\gamma_U(x,y)\to+\infty$.

Lemme 4.2.9. *Let $k>0$ and $M\geq 0$ be constants. Let $\varepsilon>0$. Assume that $(x,y)\in U\times U$ are such that*

$$\frac{k}{2\pi}|\ln\varepsilon| - M \leq \gamma_U(x,y).$$

Then there exists a constant $C=C(M,U)$ such that

$$\begin{cases}|x-y|\leq C\varepsilon^k\\ d(x,\partial U)\leq C\varepsilon^k\\ d(y,\partial U)\leq C\varepsilon^k.\end{cases}$$

Démonstration. Using relation (4.2.14) we have

$$\frac{k}{2\pi}|\ln\varepsilon| - M \leq -\frac{1}{2\pi}\ln\big(|T(x)-T(y)^*||T(y)|\big) + \frac{1}{2\pi}\ln\frac{|T(x)-T(y)|}{|x-y|}.$$

Recalling relation (4.2.15), there exists a constant C such that

$$\frac{k}{2\pi}|\ln\varepsilon| - M \leq -\frac{1}{2\pi}\ln\big(|T(x)-T(y)^*||T(y)|\big) + C,$$

and thus

$$|T(x)\overline{T(y)}-1| \leq \varepsilon^k e^{2\pi(C+M)}.$$

Moreover we have that for every $(a,b)\in D$, $|a-b|\leq|1-a\bar{b}|$. Indeed, one can check that

$$|1-a\bar{b}|^2-|a-b|^2=(1-|b|^2)(1-|a|^2)>0.$$

Therefore

$$|T(x)-T(y)|\leq\varepsilon^k e^{2\pi(C-M)}$$

and relation (4.2.15) gives that

$$|x-y|\leq C\varepsilon^k.$$

It also yields that

$$(1-|T(x)|^2)(1-|T(y)|^2)\leq|T(x)\overline{T(y)}-1|^2\leq C\varepsilon^{2k}.$$

That means that either $1-|T(x)|^2\leq\sqrt{C}\varepsilon^k$, or $1-|T(y)|^2\leq\sqrt{C}\varepsilon^k$. We can assume without loss of generality that $1-|T(x)|^2\leq\sqrt{C}\varepsilon^k$. We infer that $1-|T(x)|\leq\sqrt{C}\varepsilon^k$ and by the properties of the map T^{-1} given in Theorem 4.2.3, we conclude that $d(x,\partial U)\leq C'\varepsilon^k$. Since $|x-y|\leq C\varepsilon^k$, that means that $d(x,\partial U)\leq C\varepsilon^k$ and $d(y,\partial U)\leq C\varepsilon^k$. □

This lemma also stands in the case of an exterior domain Π as follows.

Lemme 4.2.10. *Let $k>0$ and $M\geq0$ be constants. Let $\varepsilon>0$. Let $\mathcal{U}\subset\Pi$ be a bounded domain. Assume that $(x,y)\in\mathcal{U}\times\mathcal{U}$ are such that*

$$\frac{k}{2\pi}|\ln\varepsilon|-M\leq\gamma_\Pi(x,y).$$

Then there exists a constant $C=C(M,\mathcal{U},\Pi)$ such that

$$\begin{cases}|x-y|\leq C\varepsilon^k\\ d(x,\partial\Pi)\leq C\varepsilon^k\\ d(y,\partial\Pi)\leq C\varepsilon^k.\end{cases}$$

Démonstration. We argue as in Lemma 4.2.9. Relation (4.2.8) gives that if $T:\Pi\to\Pi_D$ is a biholomorphism, then

$$\gamma_\Pi(x,y)=-\frac{1}{2\pi}\ln\big(|T(x)-T(y)^*||T(y)|\big)+\frac{1}{2\pi}\ln\frac{|T(x)-T(y)|}{|x-y|}.$$

Relation (4.2.15) holds true on the set $\overline{\mathcal{U}}\times\overline{\mathcal{U}}$, so we still have that

$$|T(x)\overline{T(y)}-1|\leq\varepsilon^k e^{2\pi(C+M)}.$$

We notice now that for every $(a,b)\in\Pi_D$ we also have that $|a-b|\leq|1-a\bar{b}|$ since $(1-|b|^2)(1-|a|^2)>0$. Thus

$$|x-y|\leq C\varepsilon^k$$

and

$$(1-|T(x)|^2)(1-|T(y)|^2) \le |T(x)\overline{T(y)}-1|^2 \le C\varepsilon^{2k}.$$

We have either $|T(x)|^2-1 \le \sqrt{C}\varepsilon^k$, or $|T(y)|^2-1 \le \sqrt{C}\varepsilon^k$, and by the same argument, recalling Theorem 4.2.4, we have that

$$|T(x)|^2 - 1 \le \sqrt{C}\varepsilon^k \Longrightarrow d(x,\partial\Pi) \le C\varepsilon^k.$$

This proves the lemma. □

The next lemma give the formula for the exterior normal vector to ∂U.

Lemme 4.2.11. *Let $x \in \partial U$. If $n_U(x)$ is the exterior normal unit vector to ∂U in x, and $n_D(T(x))$ the exterior normal unit vector to ∂D in $T(x)$, then*

$$n_U(x) = \frac{1}{|T'(x)|}\begin{pmatrix}\partial_1 T_1(x) n_D^1(T(x)) + \partial_1 T_2(x) n_D^2(T(x)) \\ \partial_1 T_1(x) n_D^2(T(x)) - \partial_1 T_2(x) n_D^1(T(x))\end{pmatrix}. \tag{4.2.17}$$

Démonstration. Since we chose U and T as in Theorem 4.2.3, the map :

$$\Gamma : \mathbb{R} \to \partial U, \quad \Gamma(\theta) = T^{-1}(e^{i\theta})$$

is well defined and smooth. Therefore, denoting $x = \Gamma(\theta)$ we have that $T(x) = e^{i\theta}$ and

$$\Gamma'(\theta) = ie^{i\theta}(T^{-1})'(e^{i\theta}) = \frac{ie^{i\theta}}{T'(T^{-1}(e^{i\theta}))} = \frac{ie^{i\theta}}{T'(x)}.$$

Naturally, the exterior normal unit vector $n_D(e^{i\theta})$ to ∂D in $e^{i\theta}$ is itself $e^{i\theta}$. Since an holomorphic map preserves the orientation, $-i\frac{\Gamma'(\theta)}{|\Gamma'(\theta)|}$ is the exterior normal unit vector to ∂U in $T^{-1}(e^{i\theta}) = x$. Therefore

$$n_U(x) = -i\frac{\Gamma'(\theta)}{|\Gamma'(\theta)|} = \frac{e^{i\theta}}{T'(x)}|T'(x)| = \frac{n_D(T(x))\overline{T'(x)}}{|T'(x)|}.$$

We then compute the product

$$n_D\overline{T'} = (n_D^1 + in_D^2)(\partial_1 T_1 - i\partial_1 T_2) = n_D^1\partial_1 T_1 + n_D^2 \partial_1 T_2 + i(n_D^2\partial_1 T_1 - n_D^1 \partial_1 T_2)$$

and the lemma is now proved. □

We extend the map n_D to the interior of D by the natural formula $n_D(x) = x$. We then extend the map n_U to U by formula (4.2.17). The following lemma holds true.

Lemme 4.2.12. *There exists a constant C such that for every $x \in U$, we have that*

$$|\nabla^\perp \widetilde{\gamma}_U(x) \cdot n_U(x)| \le C$$

Démonstration. We compute the scalar product $\nabla^\perp\widetilde{\gamma}_U \cdot n_U$ by using relations (4.2.12) and (4.2.17). We obtain

$$\nabla^\perp\widetilde{\gamma}_U \cdot n_U = \begin{pmatrix} \partial_1\widetilde{\gamma}_D \partial_1 T_2 - \partial_2\widetilde{\gamma}_D\partial_1 T_1 + \dfrac{+\partial_1 T_1 \partial_1^2 T_2 - \partial_1 T_2 \partial_1^2 T_1}{2\pi|T'|^2} \\ \\ \partial_1\widetilde{\gamma}_D\partial_1 T_1 + \partial_2\widetilde{\gamma}_D\partial_1 T_2 + \dfrac{\partial_1 T_1\partial_1^2 T_1 + \partial_1 T_2 \partial_1^2 T_2}{2\pi|T'|^2}\end{pmatrix} \cdot \frac{1}{|T'|}\begin{pmatrix}\partial_1 T_1 n_D^1 + \partial_1 T_2 n_D^2 \\ \partial_1 T_1 n_D^2 - \partial_1 T_2 n_D^1\end{pmatrix}.$$

We write

$$\nabla^\perp \widetilde{\gamma}_U \cdot n_U \equiv \frac{1}{|T'|} A + \frac{1}{2\pi |T'|^3} B$$

with

$$A = \begin{pmatrix} \partial_1 \widetilde{\gamma}_D \partial_1 T_2 - \partial_2 \widetilde{\gamma}_D \partial_1 T_1 \\ \partial_1 \widetilde{\gamma}_D \partial_1 T_1 + \partial_2 \widetilde{\gamma}_D \partial_1 T_2 \end{pmatrix} \cdot \begin{pmatrix} \partial_1 T_1 n_D^1 + \partial_1 T_2 n_D^2 \\ \partial_1 T_1 n_D^2 - \partial_1 T_2 n_D^1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} \partial_1 T_1 \partial_1^2 T_2 - \partial_1 T_2 \partial_1^2 T_1 \\ \partial_1 T_1 \partial_1^2 T_1 + \partial_1 T_2 \partial_1^2 T_2 \end{pmatrix} \cdot \begin{pmatrix} \partial_1 T_1 n_D^1 + \partial_1 T_2 n_D^2 \\ \partial_1 T_1 n_D^2 - \partial_1 T_2 n_D^1 \end{pmatrix}.$$

We have that

$$\begin{aligned} A &= n_D^1 \Big[\partial_1 \widetilde{\gamma}_D \big(\partial_1 T_1 \partial_1 T_2 - \partial_1 T_2 \partial_1 T_1 \big) - \partial_2 \widetilde{\gamma}_D \big(\partial_1 T_1 \partial_1 T_1 + \partial_1 T_2 \partial_1 T_2 \big) \Big] \\ &\quad + n_D^2 \Big[\partial_1 \widetilde{\gamma}_D \big(\partial_1 T_2 \partial_1 T_2 + \partial_1 T_1 \partial_1 T_1 \big) + \partial_2 \widetilde{\gamma}_D \big(- \partial_1 T_2 \partial_1 T_1 + \partial_1 T_1 \partial_1 T_2 \big) \Big] \\ &= -n_D^1 \partial_2 \widetilde{\gamma}_D |T'|^2 + n_D^2 \partial_1 \widetilde{\gamma}_D |T'|^2 \\ &= |T'|^2 n_D \cdot \nabla^\perp \widetilde{\gamma}_D. \end{aligned}$$

We know from relation (4.2.5) that $\widetilde{\gamma}_D$ is a radial function and thus $n_D \cdot \nabla^\perp \widetilde{\gamma}_D = 0$. So $A = 0$. We now compute B.

$$\begin{aligned} B &= \begin{pmatrix} \partial_1 T_1 \partial_1^2 T_2 - \partial_1 T_2 \partial_1^2 T_1 \\ \partial_1 T_1 \partial_1^2 T_1 + \partial_1 T_2 \partial_1^2 T_2 \end{pmatrix} \cdot \begin{pmatrix} \partial_1 T_1 n_D^1 + \partial_1 T_2 n_D^2 \\ \partial_1 T_1 n_D^2 - \partial_1 T_2 n_D^1 \end{pmatrix} \\ &= n_D^1 (\partial_1^2 T_1 (-2 \partial_1 T_2 \partial_1 T_1) + \partial_1^2 T_2 ((\partial_1 T_1)^2 - (\partial_1 T_2)^2)) \\ &\quad + n_D^2 (\partial_1^2 T_1 ((\partial_1 T_1)^2 - (\partial_1 T_2)^2) + 2 \partial_1^2 T_2 \partial_1 T_2 \partial_1 T_1) \\ &= 2\pi \psi(T) \cdot n_D, \end{aligned}$$

where the map ψ is defined by relation (4.2.13), and is bounded. Since there exists a constant m such that $|T'(x)| > m > 0$, we infer that there exists a constant C such that

$$|\nabla^\perp \widetilde{\gamma}_U(x) \cdot n_U(x)| \leq C.$$

□

Similarly, the normal vector n_{Π_D} can be extended to Π_D as a smooth function by the formula $n_{\Pi_D}(x) = -x$ for all $x \in \Pi_D$. We can then reproduce Lemma 4.2.11 to extend n_Π to the interior of Π by the formula

$$n_\Pi(x) = \frac{1}{|T'(x)|} \begin{pmatrix} \partial_1 T_1(x) n_{\Pi_D}^1 (T(x)) + \partial_1 T_2(x) n_{\Pi_D}^2 (T(x)) \\ \partial_1 T_1(x) n_{\Pi_D}^2 (T(x)) - \partial_1 T_2(x) n_{\Pi_D}^1 (T(x)) \end{pmatrix}.$$

Lemma 4.2.12 can be adapted to the exterior domain case in a straightforward manner. We obtain that for any exterior domain Π, and any bounded subset $\mathcal{U} \subset \Pi$, there exists a constant C such that $\forall x \in \mathcal{U}$,

$$|\nabla^\perp \widetilde{\gamma}_\Pi(x) \cdot n_\Pi(x)| \leq C. \tag{4.2.18}$$

4.2.5 Inequalities for simply connected bounded domains and exterior domains

We start with an estimate on the gradient of the Green's function. Let $\mathcal{U}$ be a bounded domain with $C^{2,\alpha}$ boundary. There exists a constant C depending only on $\mathcal{U}$ such that

$$\forall (x,y) \in \mathcal{U} \times \mathcal{U}, \ \ x \neq y, \quad |\nabla_x G_{\mathcal{U}}(x,y)| \leq \frac{C}{|x-y|}. \tag{4.2.19}$$

This estimate can be found in [58], see also [49, Proposition 6.1].

Now we can state the required inequalities for simply connected domains. We recall that U is a simply connected bounded domain with $C^{2,\alpha}$ boundary.

Lemme 4.2.13. *The following inequalities hold true for any $\kappa < 1$:*

$$\iint_{U\times U} \frac{1}{d(x,\partial U)^\kappa} |\nabla_x G_U(x,y)\cdot\nabla^\perp\widetilde{\gamma}_U(x)|\mathrm{d}x\mathrm{d}y < \infty$$

and

$$\iint_{U\times U} \frac{1}{|x-y|^\kappa} |\nabla_x G_U(x,y)\cdot\nabla^\perp\widetilde{\gamma}_U(x)|\mathrm{d}x\mathrm{d}y < \infty.$$

Démonstration. We start by denoting either $p(x,y) = \frac{1}{|x-y|^\kappa}$ or $p(x,y) = \frac{1}{d(x,\partial U)^\kappa}$. We use Lemma 4.2.7. Since ψ is bounded, and since from Theorem 4.2.3 there exist constants m and M such that for every $x \in U$, $0 < m < |T'(x)| < M$, we obtain that

$$\iint_{U\times U} p(x,y)|\nabla_x G_U(x,y)\cdot\nabla^\perp\widetilde{\gamma}_U(x)|\mathrm{d}x\mathrm{d}y$$
$$\le C\iint_{U\times U} p(x,y)[|\nabla_x G_D(T(x),T(y))\cdot\nabla^\perp\widetilde{\gamma}_D(T(x))| + |\nabla_x G_D(T(x),T(y))|]\mathrm{d}x\mathrm{d}y.$$

We now change variables using that $0 < m < |T'(x)| < M$, to obtain that

$$\iint_{U\times U} p(x,y)|\nabla_x G_U(x,y)\cdot\nabla^\perp\widetilde{\gamma}_U(x)|\mathrm{d}x\mathrm{d}y$$
$$\le C\iint_{D\times D} p(T^{-1}(x),T^{-1}(y))[|\nabla_x G_D(x,y)\cdot\nabla^\perp\widetilde{\gamma}_D(x)| + |\nabla_x G_D(x,y)|]\mathrm{d}x\mathrm{d}y.$$

Assume now that $p(x,y) = \frac{1}{d(x,\partial U)^\kappa}$. By the properties of the map T, there exists a constant $C > 0$ such that $d(T^{-1}(x),\partial U) \ge Cd(x,\partial D)$. We thus have that

$$p(T^{-1}(x),T^{-1}(y)) = \frac{C}{d(T^{-1}(x),\partial U)^\kappa} \le \frac{C}{d(x,\partial D)^\kappa}.$$

In the light of this inequality, using relation (4.2.19) and Lemma 4.2.1 on $\mathcal{U} = D$ yields that

$$\iint_{D\times D} p(T^{-1}(x),T^{-1}(y))|\nabla_x G_D(x,y)|\mathrm{d}x\mathrm{d}y \le C\iint_{D\times D} \frac{1}{d(x,\partial D)^\kappa|x-y|} < \infty.$$

Recalling relation (4.2.9) implies

$$\iint_{U\times U} \frac{1}{d(x,\partial U)^\kappa} |\nabla_x G_U(x,y)\cdot\nabla^\perp\widetilde{\gamma}_U(x)|\mathrm{d}x\mathrm{d}y < \infty.$$

We assume next that $p(x,y) = \frac{1}{|x-y|^\kappa}$. By the properties of the map T, we have for every $(x,y) \in D\times D$, $x \ne y$ that $|T^{-1}(x) - T^{-1}(y)| > C|x-y|$. Thus

$$p(T^{-1}(x),T^{-1}(y)) \le Cp(x,y).$$

Using once again (4.2.19) and Lemma 4.2.1, we have that

$$\iint_{D\times D} p(T^{-1}(x),T^{-1}(y))|\nabla_x G_D(x,y)|\mathrm{d}x\mathrm{d}y \le C\iint_{D\times D} \frac{1}{|x-y|^{1+\kappa}} < \infty,$$

and recalling relation (4.2.10) we conclude that

$$\iint_{U\times U} \frac{1}{|x-y|^\kappa} |\nabla_x G_U(x,y)\cdot\nabla^\perp\widetilde{\gamma}_U(x)|\mathrm{d}x\mathrm{d}y < \infty.$$

The lemma is proved. □

We now show these inequalities for Π_D. However the integral must be taken on a bounded subset.

Lemme 4.2.14. *Let $\mathcal{U}$ be any bounded subset of Π_D. The following inequalities hold true for any $\kappa < 1$:*

$$\iint_{\mathcal{U}\times\mathcal{U}} \frac{1}{d(x,\partial\Pi_D)^\kappa} |\nabla_x G_{\Pi_D}(x,y)\cdot\nabla^\perp\widetilde{\gamma}_{\Pi_D}(x)|\mathrm{d}x\mathrm{d}y < \infty$$

and

$$\iint_{\mathcal{U}\times\mathcal{U}} \frac{1}{|x-y|^\kappa} |\nabla_x G_{\Pi_D}(x,y)\cdot\nabla^\perp\widetilde{\gamma}_{\Pi_D}(x)|\mathrm{d}x\mathrm{d}y < \infty.$$

Démonstration. We recall that G_D and G_{Π_D} have the same explicit expression, see relations (4.2.3) and (4.2.8). Noticing that for $(x,y)\in\mathbb{C}\times\mathbb{C}^*$, $|x-y^*||y| = |x\overline{y}-1|$ we obtain that for every $(x,y)\in\Pi_D\times\Pi_D$, $x\neq y$,

$$G_D\left(\frac{1}{x},\frac{1}{y}\right) = \frac{1}{2\pi}\ln\frac{\left|\frac{1}{x}-\frac{1}{y}\right|}{|\frac{1}{x\overline{y}}-1|} = \frac{1}{2\pi}\ln\frac{|x-y|}{|x\overline{y}-1|} = G_{\Pi_D}(x,y).$$

We can reproduce the proof of Lemma 4.2.7 where we replace the biholomorphism $T:U\to D$ with the map $T:\Pi_D\to D$, $T(z)=1/z$. This map is holomorphic and satisfies that $0<m<|T'(x)|<M$ and $|T''(x)|<M$ on $\mathcal{U}$. The calculations given in the proof of Lemma 4.2.7 allow to obtain the following bound

$$|\nabla_x G_{\Pi_D}(x,y)\cdot\nabla^\perp\widetilde{\gamma}_{\Pi_D}(x)| \leq C\left|\nabla_x G_D\left(\frac{1}{x},\frac{1}{y}\right)\cdot\nabla^\perp\widetilde{\gamma}_D\left(\frac{1}{x}\right)\right| + C\left|\nabla_x G_D\left(\frac{1}{x},\frac{1}{y}\right)\right|.$$

for all $x,y\in\mathcal{U}$.

Thus for any $p:\mathbb{C}^2\to\mathbb{R}$, we have that

$$\begin{aligned}\iint_{\mathcal{U}\times\mathcal{U}} & p(x,y)|\nabla_x G_{\Pi_D}(x,y)\cdot\nabla^\perp\widetilde{\gamma}_{\Pi_D}(x)|\mathrm{d}x\mathrm{d}y \\ & \leq C\iint_{\mathcal{U}\times\mathcal{U}} p(x,y)\left(\left|\nabla_x G_D\left(\frac{1}{x},\frac{1}{y}\right)\cdot\nabla^\perp\widetilde{\gamma}_D\left(\frac{1}{x}\right)\right| + \left|\nabla_x G_D\left(\frac{1}{x},\frac{1}{y}\right)\right|\right)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Changing variables we obtain

$$\begin{aligned}\iint_{\mathcal{U}\times\mathcal{U}} & p(x,y)|\nabla_x G_{\Pi_D}(x,y)\cdot\nabla^\perp\widetilde{\gamma}_{\Pi_D}(x)|\mathrm{d}x\mathrm{d}y \\ & \leq C\iint_{T(\mathcal{U})\times T(\mathcal{U})} p\left(\frac{1}{x},\frac{1}{y}\right)\Big(|\nabla_x G_D(x,y)\cdot\nabla^\perp\widetilde{\gamma}_D(x)| + |\nabla_x G_D(x,y)|\Big)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

The end of the proof is very similar to Lemma 4.2.13. We start by showing that in the case $p(x,y) = \frac{1}{d(x,\partial\Pi_D)^\kappa} = \frac{1}{(|x|-1)^\kappa}$, we have for every $x,y\in T(\mathcal{U})\times T(\mathcal{U})$ that

$$p\left(\frac{1}{x},\frac{1}{y}\right) = \frac{|x|^\kappa}{(1-|x|)^\kappa} \leq \frac{1}{d(x,\partial D)^\kappa}.$$

We can use relation (4.2.19) and Lemma 4.2.1 on $T(\mathcal{U})$ which is a bounded domain, to observe that

$$\iint_{T(\mathcal{U})\times T(\mathcal{U})} \frac{1}{d(x,\partial D)^\kappa}|\nabla_x G_D(x,y)|\mathrm{d}x\mathrm{d}y < \infty.$$

Relation (4.2.9) implies that

$$\iint_{T(\mathcal{U})\times T(\mathcal{U})} \frac{1}{d(x,\partial D)^\kappa}|\nabla_x G_D(x,y)\cdot\nabla^\perp\widetilde{\gamma}_D(x)|\mathrm{d}x\mathrm{d}y < \infty.$$

We proved that

$$\iint_{\mathcal{U}\times\mathcal{U}} \frac{1}{d(x,\partial\Pi_D)^\kappa}|\nabla_x G_{\Pi_D}(x,y)\cdot\nabla^\perp\widetilde{\gamma}_{\Pi_D}(x)|\mathrm{d}x\mathrm{d}y < \infty.$$

Similarly, in the case $p(x,y) = \frac{1}{|x-y|^\kappa}$, we have that

$$p\left(\frac{1}{x},\frac{1}{y}\right) = \frac{|xy|^\kappa}{|y-x|^\kappa} \le \frac{1}{|y-x|^\kappa}.$$

Since $\mathcal{U}$ is bounded, Lemma 4.2.1 apply, and using relation (4.2.19) yields that

$$\iint_{\mathcal{U}\times\mathcal{U}} \frac{1}{|y-x|^\kappa}|\nabla_x G_D(x,y)|\mathrm{d}x\mathrm{d}y < \infty.$$

Relation (4.2.10) implies that

$$\iint_{T(\mathcal{U})\times T(\mathcal{U})} \frac{1}{|x-y|^\kappa}|\nabla_x G_D(x,y)\cdot\nabla^\perp\widetilde{\gamma}_D(x)|\mathrm{d}x\mathrm{d}y < \infty.$$

We conclude that

$$\iint_{\mathcal{U}\times\mathcal{U}} \frac{1}{|x-y|^\kappa}|\nabla_x G_{\Pi_D}(x,y)\cdot\nabla^\perp\widetilde{\gamma}_{\Pi_D}(x)|\mathrm{d}x\mathrm{d}y < \infty.$$

The lemma is now proved. □

We conclude this section with the same inequalities for any exterior domain Π.

Lemme 4.2.15. *Let $\mathcal{U}$ be any bounded subset of Π. The following inequalities hold true for any $\kappa < 1$:*

$$\iint_{\mathcal{U}\times\mathcal{U}} \frac{1}{d(x,\partial\Pi)^\kappa}|\nabla_x G_{\Pi}(x,y)\cdot\nabla^\perp\widetilde{\gamma}_{\Pi}(x)|\mathrm{d}x\mathrm{d}y < \infty$$

and

$$\iint_{\mathcal{U}\times\mathcal{U}} \frac{1}{|x-y|^\kappa}|\nabla_x G_{\Pi}(x,y)\cdot\nabla^\perp\widetilde{\gamma}_{\Pi}(x)|\mathrm{d}x\mathrm{d}y < \infty.$$

Démonstration. The proof follows the same outline as the proofs of Lemmas 4.2.13 and 4.2.14. □

4.3 Multiply connected domain

We work now with the multiply connected domain Ω. There exists a compact set K such that $\Omega\setminus K$ has exactly $n+1$ connected components $V_0,\dots,V_n$ that satisfy $d(V_i,\Gamma_j) > 0$ for every $i \neq j$. For example one can take V_i to be the ε-neighborhood of Γ_i, for ε small enough. Thus we also have that

$$\Omega = K \cup \left(\bigcup_{j=0}^{m} V_j\right). \tag{4.3.1}$$

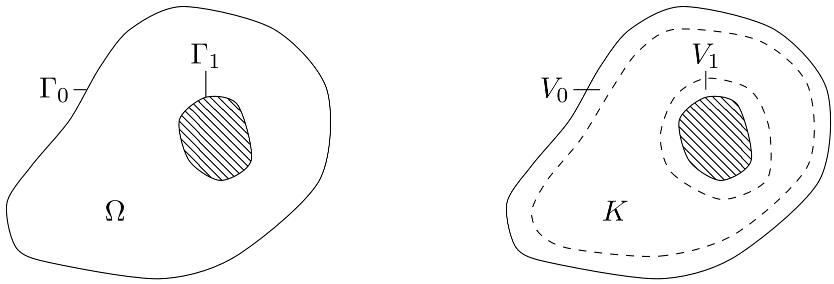

FIGURE 4.2 – Decomposition (4.3.1).

4.3.1 Biot-Savart law

Let ω be a fixed function on Ω. Obtaining the velocity u in terms of the vorticity ω is solving the following problem

$$\begin{cases} \operatorname{curl} u = \omega, & \text{in } \Omega \\ \nabla \cdot u = 0, & \text{in } \Omega \\ u \cdot n = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.3.2}$$

As in the simply connected case, a particular solution is given by

$$u(x) = \int_\Omega \nabla_x^\perp G_\Omega(x,y)\omega(y)\mathrm{d}y.$$

Since (4.3.2) is linear, the general solution is given by this particular solution plus the general solution of the homogeneous problem. The solution is of the form (see [49])

$$u(x,t) = \int \nabla_x^\perp G_\Omega(x,y)\omega(y,t)\mathrm{d}y + \sum_{j=1}^m c_{j,\omega}(t)\beta_j(x)$$

where

$$c_{j,\omega}(t) = \int w_j(x)\omega(x,t)\mathrm{d}x + \xi_j,$$

ξ_j is the circulation of the velocity u on Γ_j, $w_j : \Omega \mapsto \mathbb{R}$ are the harmonic measures defined by

$$\begin{cases} \Delta w_j = 0 & \text{in } \Omega \\ w_j = \delta_{j,l} & \text{on } \Gamma_l, 0 \le l \le n. \end{cases}$$

and β_j are a basis of the harmonic vector fields defined by

$$\begin{cases} \nabla \cdot \beta_j = 0 & \text{in } \Omega \\ \operatorname{curl} \beta_j = 0 & \text{in } \Omega \\ \beta_j \cdot n = 0 & \text{in } \partial\Omega \\ \displaystyle\int_{\Gamma_\ell} \beta_j \cdot (-n^\perp)\mathrm{d}s = \delta_{j,\ell} & \forall 1 \le \ell \le m. \end{cases}$$

In the case of a discrete vorticity

$$\omega(t) = \sum_{j=1}^N a_j \delta_{x_j(t)}$$

we define the point vortex dynamics in multiply connected bounded domains as follows :

$$\forall 1 \leq i \leq N, \quad \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = \sum_{\substack{j=1 \\ j\neq i}}^{N} \nabla_x^{\perp} G_\Omega(x_i(t), x_j(t)) a_j + \nabla_x^{\perp} \gamma_\Omega(x_i(t), x_i(t)) a_i + \sum_{j=1}^{m} c_j(t)\beta_j(x)$$

where

$$c_j(t) = \xi_j + \sum_{k=1}^{N} a_k w_j(x_k(t)).$$

Let us observe that by the Kelvin theorem, the circulations ξ_j are constant in time. They are therefore prescribed at the initial time. The harmonic measures w_j being smooth, we observe that the functions $c_j : \mathbb{R}^+ \to \mathbb{R}$ are bounded.

4.3.2 Inequalities for multiply connected domains

We know from [49, Proposition 6.1] that the inequality

$$|G_\Omega(x,y)| \leq C(1 + |\ln|x-y||) \tag{4.3.3}$$

holds true for bounded domains. We also recall relation (4.2.19) :

$$|\nabla_x G_\Omega(x,y)| \leq \frac{C}{|x-y|}.$$

We combine this with Lemma 4.2.1. Since Ω is bounded, for any $\kappa < 1$ we have that

$$\iint_{\Omega\times\Omega} \frac{1}{|x-y|^\kappa} |\nabla_x G_\Omega(x,y)| \mathrm{d}x\mathrm{d}y < \infty \tag{4.3.4}$$

and

$$\iint_{\Omega\times\Omega} \frac{1}{d(x,\partial\Omega)^\kappa} |\nabla_x G_\Omega(x,y)| \mathrm{d}x\mathrm{d}y < \infty. \tag{4.3.5}$$

The following proposition gives an estimate of the map $\widetilde{\gamma}_\Omega$ near the boundary.

Proposition 4.3.1 (Gustafsson [40] Proposition 3.3)**.** *Denoting by K_j the connected components of $\mathbb{R}^2 \setminus \Omega$ and $d_j(x) = \inf\{|x-y|, y \in K_j\}$, $D_j(x) = \sup\{|x-y|, y \in K_j\}$, we have that*

$$\forall x \in \Omega, \quad \ln d(x,\partial\Omega) \leq -2\pi\widetilde{\gamma}_\Omega(x) \leq \min_j \ln \frac{4d_j(x)}{1 - \frac{d_j(x)}{D_j(x)}}.$$

Clearly, $d_j(x) < D_j(x)$ for every $x \in \Omega$, so by compactness

$$\sup\left\{\frac{d_j(x)}{D_j(x)}, x \in \Omega\right\} < 1.$$

That means that there exists a constant C_1 depending only on Ω such that for every $x \in \Omega$

$$\ln d(x,\partial\Omega) \leq -2\pi\widetilde{\gamma}_\Omega(x) \leq \min_j \ln d_j + C_1$$

and thus

$$\ln d(x,\partial\Omega) \leq -2\pi\widetilde{\gamma}_\Omega(x) \leq \ln d(x,\partial\Omega) + C_1. \tag{4.3.6}$$

In particular,

$$\widetilde{\gamma}_\Omega(x) \sim -\frac{1}{2\pi} \ln d(x,\partial\Omega) \quad \text{as } x \to \partial\Omega$$

and

$$\inf_\Omega \widetilde{\gamma}_\Omega = \min_\Omega \widetilde{\gamma}_\Omega > -\infty. \tag{4.3.7}$$

In addition, we state in the following proposition an estimate of $\nabla\widetilde{\gamma}_\Omega$ near the boundary.

Proposition 4.3.2 (Gustafsson [40] Proposition 3.5)**.** *There exists a constant C such that for every $x \in \Omega$,*

$$|\nabla \widetilde{\gamma}_\Omega(x)| \leq \frac{C}{d(x, \partial\Omega)} \tag{4.3.8}$$

Moreover, we can take $C = \frac{1}{2\pi}$ if Ω is simply connected.

We now want to compare the map γ_Ω near the boundary Γ_j, to the map γ_{Ω_j}.

Lemme 4.3.3. *For any $0 \leq j \leq m$, the map $\gamma_\Omega - \gamma_{\Omega_j}$ is bounded in $V_j \times \Omega$.*

Démonstration. We have that

$$\gamma_\Omega(x, y) - \gamma_{\Omega_j}(x, y) = G_\Omega(x, y) - G_{\Omega_j}(x, y)$$

We fix $x \in V_j$, and we define $F(y) = G_\Omega(x, y) - G_{\Omega_j}(x, y)$. It satisfies that

$$\begin{cases} \Delta_y F(y) = 0, & \text{on } \Omega, \\ F(y) = 0, & \text{on } \Gamma_j, \\ |F(y)| \leq C & \text{on } \Gamma_k,\ k \neq j, \end{cases}$$

where

$$C = \sup_{\substack{x \in V_j \\ k \neq j \\ y \in \Gamma_k}} |G_{\Omega_j}(x, y)| < \infty$$

is a constant that does not depend on $x \in V_j$. The supremum is finite since $d(V_j, \Gamma_k) > 0$ for each $k \neq j$. Therefore by the maximum principle,

$$|\gamma_\Omega(x, y) - \gamma_{\Omega_j}(x, y)| = |F(y)| \leq \max_{y \in \partial\Omega} |F(y)| \leq C$$

for every $(x, y) \in V_j \times \Omega$. □

Observe that, for $x \in V_j$, the map $\widetilde{F}(y) = \nabla_x \gamma_\Omega(x, y) - \nabla_x \gamma_{\Omega_j}(x, y)$ satisfies the exact same problem : its Laplacian vanishes over Ω, $\widetilde{F}(y) = 0$ on Γ_j and it is bounded on Γ_k by a map $C(x)$ that is itself bounded in V_j. Hence the map $\nabla_x \gamma_\Omega - \nabla_x \gamma_{\Omega_j}$ is also bounded in $V_j \times \Omega$. Since $V_j \times V_j \subset V_j \times \Omega$, we can set $y = x$ in those inequalities and obtain similar bounds for $\widetilde{\gamma}_\Omega - \widetilde{\gamma}_{\Omega_j}$ and $\nabla \widetilde{\gamma}_\Omega - \nabla \widetilde{\gamma}_{\Omega_j}$. In this manner we obtain the following corollary.

Corollaire 4.3.4. *For any $0 \leq j \leq m$, we have that*
- *the map $\nabla_x \gamma_\Omega - \nabla_x \gamma_{\Omega_j}$ is bounded in $V_j \times \Omega$.*
- *the map $\widetilde{\gamma}_\Omega - \widetilde{\gamma}_{\Omega_j}$ is bounded in V_j.*
- *the map $\nabla \widetilde{\gamma}_\Omega - \nabla \widetilde{\gamma}_{\Omega_j}$ is bounded in V_j.*

Lemma 4.2.8 is stated for bounded simply-connected domains, it is also true for exterior domains, see relation (4.2.16). Lemma 4.3.3 allows to prove it for multiply connected domains.

Corollaire 4.3.5. *We have that for any $x_0 \in \partial\Omega$, $\gamma_\Omega(x, y) \underset{x,y \to x_0 \in \partial\Omega}{\longrightarrow} +\infty$.*

We now combine Lemma 4.2.12 and Lemma 4.3.4 to obtain the following result.

Corollaire 4.3.6. *There exists a constant C such that for every $x \in \Omega$ and every $1 \leq j \leq m$,*

$$|\nabla \widetilde{\gamma}_\Omega(x) \cdot \beta_j(x)| \leq C.$$

Démonstration. Let $1 \leq j \leq m$ and $0 \leq k \leq m$. In a neighborhood of Γ_k, we decompose

$$\beta_j(x) \equiv \beta_j^1(x) n_{\Omega_k}(x) + \beta_j^2(x) n_{\Omega_k}^{\perp}(x).$$

Since β_j is tangent to the boundary, $\beta_j^1(x) = 0$ when $x \in \Gamma_k$. Since β_j is smooth, there exists a constant $C_{j,k}$ such that $|\beta_j^1(x)| \leq C_{j,k} d(x, \Gamma_k)$ in a neighborhood of Γ_k. Recalling relation (4.3.8) and provided that the neighborhood is sufficiently small so that $d(x, \partial\Omega) = d(x, \Gamma_k)$, we have that

$$|\beta_j^1(x) \nabla \widetilde{\gamma}_\Omega(x) \cdot n_{\Omega_k}(x)| \leq C_{j,k} d(x, \Gamma_k) \frac{C}{d(x, \partial\Omega)} \leq C_{j,k}.$$

If $k = 0$, we apply Lemma 4.2.12. If $k \neq 0$, we use relation (4.2.18). In both cases, it yields that

$$|\nabla^{\perp} \widetilde{\gamma}_{\Omega_k}(x) \cdot n_{\Omega_k}(x))| \leq C_k$$

in a neighborhood of Γ_k. Thus by Corollary 4.3.4, and since n_{Ω_k} is bounded in that neighborhood,

$$|\nabla^{\perp} \widetilde{\gamma}_\Omega(x) \cdot n_{\Omega_k}(x))| \leq C_k.$$

Consequently, since β_j is bounded,

$$|\beta_j^2(x) \nabla \widetilde{\gamma}_\Omega(x) \cdot n_{\Omega_k}^{\perp}(x))| \leq C_{j,k}.$$

Therefore on this neighborhood of the boundary Γ_k, there exists a constant $C_{j,k}$ such that

$$|\nabla^{\perp} \widetilde{\gamma}_\Omega(x) \cdot \beta_j(x)| \leq C_{j,k}$$

Outside of each of these neighborhoods, we know that the maps $\nabla \widetilde{\gamma}_\Omega$ and β_j are bounded. Therefore, as there are a finite number of boundaries Γ_k, and of maps β_j, there exists a constant C depending only on Ω such that for every $x \in \Omega$,

$$|\nabla^{\perp} \widetilde{\gamma}_\Omega(x) \cdot \beta_j(x)| \leq C.$$

□

We can now extend Lemma 4.2.13 to the case of multiply connected domains.

Lemme 4.3.7. *The following inequalities hold true for any $\kappa < 1$:*

$$\iint_{\Omega \times \Omega} \frac{1}{|x-y|^\kappa} |\nabla_x G_\Omega(x,y) \cdot \nabla^{\perp} \widetilde{\gamma}_\Omega(x)| \mathrm{d}x \mathrm{d}y < \infty$$

and

$$\iint_{\Omega \times \Omega} \frac{1}{d(x, \partial\Omega)^\kappa} |\nabla_x G_\Omega(x,y) \cdot \nabla^{\perp} \widetilde{\gamma}_\Omega(x)| \mathrm{d}x \mathrm{d}y < \infty.$$

Démonstration. Let us introduce the map h defined by

$$h(x,y) = \nabla_x G_\Omega(x,y) \cdot \nabla^{\perp} \widetilde{\gamma}_\Omega(x).$$

We must show that $ph \in L^1(\Omega \times \Omega)$ for $p(x,y) = \frac{1}{|x-y|^\kappa}$ and also for $p(x,y) = \frac{1}{d(x,\partial\Omega)^\kappa}$. We split the integral using the decomposition (4.3.1), pictured in Figure 4.2.

First, there exists a constant C such that $|\nabla^{\perp} \widetilde{\gamma}_\Omega(x)| \leq C$ on K. Relations (4.3.4) and (4.3.5) thus imply that $ph \in L^1(K \times \Omega)$ for both expression of p. Now we must prove that $ph \in L^1(V_j \times \Omega)$ for every $0 \leq j \leq m$. We fix $0 \leq j \leq m$. By Corollary 4.3.4 as well as relations (4.3.4) and (4.3.5), we know that proving $ph \in L^1(V_j \times \Omega)$ is equivalent to proving that $ph_1 \in L^1(V_j \times \Omega)$, with

$$h_1(x,y) = \nabla_x G_\Omega(x,y) \cdot \nabla^{\perp} \widetilde{\gamma}_{\Omega_j}(x).$$

Let us introduce
$$h_2(x,y) = \nabla_x G_{\Omega_j}(x,y)\cdot\nabla^\perp\widetilde{\gamma}_{\Omega_j}(x).$$
We have that $h_1 - h_2 \in C^\infty(\Omega\times\Omega)$ and that $\Delta_y(h_1-h_2) = 0$. The maximum principle yields

$$\forall (x,y)\in V_j\times\Omega,\ \ |h_1(x,y)-h_2(x,y)| \le \sup_{y\in\partial\Omega}|h_1(x,y)-h_2(x,y)|. \tag{4.3.9}$$

However, $h_1(x,y) = 0$ when $y\in\partial\Omega$, since $\forall x\in\Omega, G_\Omega(x,y) = 0$ when $y\in\partial\Omega$. And similarly, $h_2(x,y) = 0$ when $y\in\partial\Omega_j$. Thus,

$$\sup_{y\in\partial\Omega}|h_1(x,y)-h_2(x,y)| = \sup_{\substack{k\neq j\\ y\in\Gamma_k}}|h_2(x,y)|.$$

We need to bound $h_2(x,y)$ when $x\in V_j$ and $y\in\Gamma_k$. We decompose

$$\nabla_x G_{\Omega_j}(x,y) \equiv g_1(x,y)n_{\Omega_j(x)} + g_2(x,y)n^\perp_{\Omega_j(x)}$$

where $n_{\Omega_j(x)}$ is defined in Section 4.2.4. We have that $g_2(x,y) = 0$ when $x\in\Gamma_j$ since $G_{\Omega_j}(x,y) = 0$ for every $(x,y)\in\Gamma_j\times\Gamma_k$ so $\nabla_x G_{\Omega_j}(x,y)$ is normal to the boundary Γ_j. By Theorem 4.2.6, G_{Ω_j} is C^2 up to the boundary except on the diagonal, thus there exists a constant C independent of $y\in\Gamma_k$ such that $|g_2(x,y)|\le Cd(x,\Gamma_j)$ for all $x\in V_j$ and $y\in\Gamma_k$. Using relation (4.3.8), we have that
$$|g_2(x,y)n^\perp_{\Omega_j(x)}\cdot\nabla^\perp\widetilde{\gamma}_{\Omega_j}(x)|\le C.$$
Using Lemma 4.2.12 if $j=0$ and relation (4.2.18) if $j\neq 0$, we have that

$$|g_1(x,y)n_{\Omega_j(x)}\cdot\nabla^\perp\widetilde{\gamma}_{\Omega_j}(x)|\le C$$

for all $x\in V_j$ and $y\in\Gamma_k$. So there exists a constant independent of $x\in V_j$ and $y\in\Gamma_k$ such that $|h_2(x,y)|\le C$. Thus there exists a constant C such that

$$\sup_{y\in\partial\Omega}|h_1(x,y)-h_2(x,y)|\le C. \tag{4.3.10}$$

Therefore, since $p\in L^1(\Omega\times\Omega)$, relations (4.3.9) and (4.3.10) yield that $p(h_1-h_2)\in L^1(V_j\times\Omega)$.

If $j=0$, we now apply Lemma 4.2.13 to Ω_0. If $j\neq 0$ we apply Lemma 4.2.15 to the domain Ω_j and to its bounded subset Ω. In both cases, we get that ph_2 is integrable on $\Omega\times\Omega$. Therefore $ph_1\in L^1(V_j\times\Omega)$ and thus $ph\in L^1(V_j\times\Omega)$. This completes the proof of the lemma. □

4.4 Completion of the proof of Theorem 4.1.2

In this section we denote by G, γ and $\widetilde{\gamma}$ the maps associated to the domain Ω in order to lighten the notations as there should be no ambiguity.

4.4.1 Construction of a regularized dynamic

We need to construct a dynamics that is well defined for every time and which is the same as the point vortex dynamics as long as no point vortex is close to the boundary and no two point vortices are close to each other. More precisely, we need to construct two functions G_ε and $\widetilde{\gamma}_\varepsilon$ such that the dynamics

$$\forall 1\le i\le N,\ \ \frac{\mathrm{d}x_i^\varepsilon(t)}{\mathrm{d}t} = \sum_{\substack{j=1\\ j\neq i}}^N \nabla_x^\perp G_\varepsilon(x_i^\varepsilon(t),x_j^\varepsilon(t))a_j + \frac{1}{2}\nabla^\perp\widetilde{\gamma}_\varepsilon(x_i^\varepsilon(t))a_i + \sum_{j=1}^m c_j(t)\beta_j(x_i^\varepsilon(t)) \tag{4.4.1}$$

is well defined in $\overline{\Omega}$ for every time. It suffices that $G_\varepsilon \in C^2(\overline{\Omega} \times \overline{\Omega})$ and $\widetilde{\gamma}_\varepsilon \in C^1(\overline{\Omega})$ and that $\nabla_x^\perp G_\varepsilon$ and $\nabla^\perp \widetilde{\gamma}_\varepsilon$ are tangent to $\partial\Omega$ when the first variable is at the boundary. Moreover, we want to ensure that the following implication is true for every $(x,y) \in \Omega \times \Omega$,

$$\begin{cases} |G_{\mathbb{R}^2}(x,y)| < \dfrac{1}{2\pi}|\ln \varepsilon| \\ |\widetilde{\gamma}(x)| < \dfrac{1}{2\pi}|\ln \varepsilon| \\ |\gamma(x,y)| < \dfrac{1}{2\pi}|\ln \varepsilon| \end{cases} \implies \begin{cases} G_\varepsilon(x,y) = G(x,y) \\ \widetilde{\gamma}_\varepsilon(x) = \widetilde{\gamma}(x). \end{cases} \tag{4.4.2}$$

This ensures that the maps G_ε, and $\widetilde{\gamma}_\varepsilon$ are good approximations of the maps G and $\widetilde{\gamma}$ when ε goes to 0.

In order to have a proper control over the regularized maps, we also want to ensure that

$$\begin{cases} |\widetilde{\gamma}_\varepsilon(x)| \leq |\widetilde{\gamma}(x)| \\ |\nabla\widetilde{\gamma}_\varepsilon(x)| \leq |\nabla\widetilde{\gamma}(x)| \\ |G_\varepsilon(x,y)| \leq |G(x,y)| \\ |\nabla_x G_\varepsilon(x,y)| \leq \dfrac{C}{|x-y|} \end{cases} \tag{4.4.3}$$

for a constant C independent of ε.

We consider $f_\varepsilon \in C^\infty(\mathbb{R},\mathbb{R})$ an odd map such that

$$\begin{cases} f_\varepsilon(r) = r, & \forall |r| < \frac{1}{2\pi}|\ln\varepsilon| \\ f_\varepsilon(r) = L_\varepsilon, & \forall r > \frac{1}{2\pi}|\ln\varepsilon| + 1 \\ 0 \leq f_\varepsilon'(r) \leq 1, & \forall r \in \mathbb{R} \end{cases}$$

for some constant L_ε.

Construction of $\widetilde{\gamma}_\varepsilon$

We define the regularized Robin function as

$$\widetilde{\gamma}_\varepsilon(x) = f_\varepsilon(\widetilde{\gamma}(x))$$

so that

$$\nabla\widetilde{\gamma}_\varepsilon(x) = \nabla\widetilde{\gamma}(x) f_\varepsilon'(\widetilde{\gamma}(x)).$$

We recall that according to Proposition 4.3.1, $\widetilde{\gamma}(x) \underset{x\to\partial\Omega}{\longrightarrow} +\infty$ and thus $\widetilde{\gamma}_\varepsilon(x) \underset{x\to\partial\Omega}{\longrightarrow} L_\varepsilon$ and $\nabla\widetilde{\gamma}_\varepsilon(x) \underset{x\to\partial\Omega}{\longrightarrow} 0$. Therefore $\widetilde{\gamma}_\varepsilon \in C^1(\overline{\Omega})$ and $\nabla^\perp\widetilde{\gamma}_\varepsilon$ is indeed tangent to the boundary since it vanishes at the boundary, and satisfies both

$$|\widetilde{\gamma}_\varepsilon(x)| \leq |\widetilde{\gamma}(x)|$$

and

$$|\nabla\widetilde{\gamma}_\varepsilon(x)| \leq |\nabla\widetilde{\gamma}(x)|.$$

Construction of G_ε

We define the regularized Green's function for $(x,y) \in \overline{\Omega} \times \overline{\Omega}$ as follows :

$$\begin{cases} G_\varepsilon(x,y) = f_\varepsilon(G_{\mathbb{R}^2}(x,y)) + f_\varepsilon(\gamma(x,y)) & \text{if } (x,y) \in \Omega\times\Omega,\ x \neq y \\ G_\varepsilon(x,y) = 0 & \text{if } x \in \partial\Omega \text{ or } y \in \partial\Omega \\ G_\varepsilon(x,x) = -L_\varepsilon + f_\varepsilon(\widetilde{\gamma}(x)) & \text{if } x \in \Omega. \end{cases}$$

Let us notice straight away that $G_\varepsilon(x,y) = G_\varepsilon(y,x)$.

We collect some properties of G_ε in the following lemma.

Lemme 4.4.1. *We have that $G_\varepsilon \in C^2(\overline{\Omega} \times \overline{\Omega})$ and that $\nabla_x^\perp G_\varepsilon(x,y)$ is tangent to $\partial\Omega$ when $x \in \partial\Omega$. Moreover, $|G_\varepsilon(x,y)| \leq |G(x,y)|$ and there exists a constant C independent of ε such that $|\nabla_x G_\varepsilon(x,y)| \leq \frac{C}{|x-y|}$.*

Démonstration. We start by proving that $G_\varepsilon \in C^1(\overline{\Omega} \times \overline{\Omega})$. Since $f_\varepsilon \in C^\infty(\mathbb{R})$, the map G_ε is clearly C^∞ on the set $\Omega \times \Omega \setminus \{x = y\}$.

We show first the continuity over $\overline{\Omega} \times \overline{\Omega}$. From relation (4.3.6) we clearly have that $G_\varepsilon(x,x) \to 0$ as $x \to \partial\Omega$, so the restriction of G_ε to the set $\partial\Omega \times \overline{\Omega} \cup \overline{\Omega} \times \partial\Omega \cup \{(x,x)\ ;\ x \in \Omega\}$ is continuous.

Let $(x_0,y_0) \in \overline{\Omega} \times \overline{\Omega}$. We take $x \to x_0$ and $y \to y_0$ and we want to show that $G_\varepsilon(x,y) \to G_\varepsilon(x_0,y_0)$. We can assume without loss of generality that $x \neq y$ and $x,y \in \Omega$. We consider several cases depending on the location of (x_0,y_0).

Assume first that $x_0 \in \partial\Omega$ and $y_0 \in \overline{\Omega}$, with $x_0 \neq y_0$. By Theorem 4.2.6,

$$G_{\mathbb{R}^2}(x,y) + \gamma(x,y) = G(x,y) \underset{(x,y)\to(x_0,y_0)}{\longrightarrow} G(x_0,y_0) = 0$$

since $x_0 \in \partial\Omega$. Moreover

$$G_{\mathbb{R}^2}(x,y) \underset{(x,y)\to(x_0,y_0)}{\longrightarrow} \frac{1}{2\pi} \ln|x_0 - y_0|$$

so

$$\gamma(x,y) \underset{(x,y)\to(x_0,y_0)}{\longrightarrow} -\frac{1}{2\pi} \ln|x_0 - y_0|.$$

We recall that f_ε is odd and continuous and thus

$$G_\varepsilon(x,y) = f_\varepsilon(G_{\mathbb{R}^2}(x,y)) + f_\varepsilon(\gamma(x,y)) \underset{(x,y)\to(x_0,y_0)}{\longrightarrow} 0 = G_\varepsilon(x_0,y_0).$$

Now assume $x_0 = y_0 \in \Omega$. Then

$$G_{\mathbb{R}^2}(x,y) \underset{(x,y)\to(x_0,y_0)}{\longrightarrow} -\infty$$

and

$$\gamma(x,y) \underset{(x,y)\to(x_0,y_0)}{\longrightarrow} \widetilde{\gamma}(x_0)$$

thus

$$G_\varepsilon(x,y) \underset{(x,y)\to(x_0,y_0)}{\longrightarrow} -L_\varepsilon + f_\varepsilon(\widetilde{\gamma}(x_0)) = G_\varepsilon(x_0,y_0).$$

Finally, assume that $x_0 = y_0 \in \partial\Omega$. Then

$$G_{\mathbb{R}^2}(x,y) \underset{(x,y)\to(x_0,y_0)}{\longrightarrow} -\infty$$

and by Corollary 4.3.5 we have that

$$\gamma(x,y) \underset{(x,y)\to(x_0,y_0)}{\longrightarrow} +\infty$$

thus

$$G_\varepsilon(x,y) \underset{(x,y)\to(x_0,y_0)}{\longrightarrow} -L_\varepsilon + L_\varepsilon = 0 = G_\varepsilon(x_0,y_0).$$

We conclude that G_ε is continuous.

We prove now that G_ε is C^1 up to the boundary. Let us compute its gradient in the first variable for any $(x,y) \in \Omega \times \Omega, x \neq y$:

$$\nabla_x G_\varepsilon(x,y) = \nabla_x G_{\mathbb{R}^2}(x,y) f'_\varepsilon(G_{\mathbb{R}^2}(x,y)) + \nabla_x \gamma(x,y) f'_\varepsilon(\gamma(x,y)). \tag{4.4.4}$$

Since f'_ε is compactly supported, $f'_\varepsilon(G_{\mathbb{R}^2}(x,y)) = 0$ in a neighborhood of the diagonal of $\overline{\Omega}\times\overline{\Omega}$. Similarly, if $x_0 \in \partial\Omega$ then $f'_\varepsilon(\gamma(x,y)) = 0$ in a neighborhood of (x_0,x_0) so $f'_\varepsilon(\gamma(x,y))$ is smooth in a neighborhood of the diagonal of $\overline{\Omega}\times\overline{\Omega}$.

Let now $x_0 \in \partial\Omega$ and $y_0 \in \overline{\Omega}$ with $x_0 \neq y_0$. Consider $x \to x_0$, $y \to y_0$ where $x \neq y$ and $x,y \in \Omega$. By Theorem 4.2.6, $\nabla_x G(x,y)$ converges, so $\nabla_x\gamma(x,y)$ converges too, and thus all quantities involved in (4.4.4) converge. We proved that $G_\varepsilon \in C^1(\overline{\Omega}\times\overline{\Omega})$. The proof that $G_\varepsilon \in C^2(\overline{\Omega}\times\overline{\Omega})$ follows along the same lines.

We now notice that $\nabla_x^\perp G_\varepsilon(x,y)$ is tangent to the boundary when $(x,y) \in \partial\Omega\times\overline{\Omega}$, since $G_\varepsilon(x,y) = 0$ when $x \in \partial\Omega$, for every $y \in \overline{\Omega}$.

We now prove the bounds stated in the lemma. Since f_ε is an odd Lipschitz map with Lipschitz constant 1, we have that

$$\forall (a,b) \in \mathbb{R}^2, |f_\varepsilon(x) + f_\varepsilon(y)| = |f_\varepsilon(x) - f_\varepsilon(-y)| \leq |x - (-y)| = |x+y|$$

and therefore

$$|G_\varepsilon(x,y)| = |f_\varepsilon(G_{\mathbb{R}^2}(x,y)) + f_\varepsilon(\gamma(x,y))| \leq |G_{\mathbb{R}^2}(x,y) + \gamma(x,y)| = |G(x,y)|.$$

Combining relation (4.1.3) with inequality (4.2.19) we have that $|\nabla_x\gamma(x,y)| \leq \frac{C}{|x-y|}$ which gives that

$$|\nabla_x G_\varepsilon(x,y)| \leq \frac{1}{2\pi|x-y|} + \frac{C}{|x-y|}.$$

Therefore $|\nabla_x G_\varepsilon(x,y)| \leq \frac{C}{|x-y|}$ and this completes the proof of the lemma. □

Note that by construction, the implication (4.4.2) is true. We thus constructed a suitable regularized dynamics.

Additional properties

We need to establish that the estimates of Lemma 4.3.7 also hold true for the regularized dynamics.

Lemme 4.4.2. *We have that for any $\kappa < 1$*

$$\iint_{\Omega\times\Omega} \frac{1}{|x-y|^\kappa} |\nabla_x G_\varepsilon(x,y)\cdot\nabla^\perp\widetilde{\gamma}_\varepsilon(x)| \mathrm{d}x\mathrm{d}y < C.$$

$$\iint_{\Omega\times\Omega} \frac{1}{d(x,\partial\Omega)^\kappa} |\nabla_x G_\varepsilon(x,y)\cdot\nabla^\perp\widetilde{\gamma}_\varepsilon(x)| \mathrm{d}x\mathrm{d}y < C.$$

where the constant C doesn't depend on ε.

Démonstration. One can check from (4.4.4) that the following relation holds true

$$\nabla_x G_\varepsilon(x,y) = \nabla_x G(x,y) f'_\varepsilon(G_{\mathbb{R}^2}(x,y)) + \nabla_x\gamma(x,y)(f'_\varepsilon(\gamma(x,y)) - f'_\varepsilon(G_{\mathbb{R}^2}(x,y))).$$

We use the expression of $\widetilde{\gamma}_\varepsilon$ and the previous relation to obtain that

$$\begin{aligned}\nabla_x G_\varepsilon(x,y)\cdot\nabla^\perp\widetilde{\gamma}_\varepsilon(x) &= \nabla_x G(x,y)\cdot\nabla_x^\perp\widetilde{\gamma}(x) f'_\varepsilon(\widetilde{\gamma}(x)) f'_\varepsilon(G_{\mathbb{R}^2}(x,y)) \\ &\quad + \nabla_x\gamma(x,y)\cdot\nabla_x^\perp\widetilde{\gamma}(x) f'_\varepsilon(\widetilde{\gamma}(x))(f'_\varepsilon(\gamma(x,y)) - f'_\varepsilon(G_{\mathbb{R}^2}(x,y))) \\ &\equiv A_{1,\varepsilon}(x,y) + A_{2,\varepsilon}(x,y).\end{aligned}$$

Recalling that $|f'_\varepsilon| \leq 1$, we can apply directly Lemma 4.3.7 to the term $A_{1,\varepsilon}$ to obtain that there exists a constant C that doesn't depend on ε such that

$$\iint_{\Omega\times\Omega} \frac{1}{d(x,\partial\Omega)^\kappa} |A_{1,\varepsilon}(x,y)| \mathrm{d}x\mathrm{d}y < C$$

and

$$\iint_{\Omega\times\Omega} \frac{1}{|x-y|^\kappa}|A_{1,\varepsilon}(x,y)|\mathrm{d}x\mathrm{d}y < C.$$

It remains to prove the same bounds for $A_{2,\varepsilon}$. Let

$$E = \{(x,y)\in\Omega\times\Omega,\ \ f'_\varepsilon(\widetilde{\gamma}(x))\big(f'_\varepsilon(\gamma(x,y)) - f'_\varepsilon(G_{\mathbb{R}^2}(x,y))\big) \neq 0\}.$$

Since $\left|f'_\varepsilon(\widetilde{\gamma}(x))\big(f'_\varepsilon(\gamma(x,y)) - f'_\varepsilon(G_{\mathbb{R}^2}(x,y))\big)\right| \le 2$, we have that

$$\iint_{\Omega\times\Omega} p(x,y)|A_{2,\varepsilon}(x,y)|\mathrm{d}x\mathrm{d}y \le \iint_E 2p(x,y)|\nabla_x\gamma(x,y)||\nabla_x^\perp\widetilde{\gamma}(x)|\mathrm{d}x\mathrm{d}y \tag{4.4.5}$$

with $p(x,y) = \frac{1}{d(x,\partial\Omega)^\kappa}$ or $p(x,y)=\frac{1}{|x-y|^\kappa}$.

We now want to show that for every $(x,y)\in E$, we have that $d(x,\partial\Omega)\ge C\varepsilon$. By construction of f_ε, if $\widetilde{\gamma}(x) > \frac{1}{2\pi}|\ln\varepsilon| + 1$ then $f'_\varepsilon(\widetilde{\gamma}(x)) = 0$. By construction of E this means that for ε small enough such that $-(\frac{1}{2\pi}|\ln\varepsilon|+1) < \min_\Omega\widetilde{\gamma}$, for every $(x,y)\in E$, we have that $|\widetilde{\gamma}(x)|\le\frac{1}{2\pi}|\ln\varepsilon|+1$. Moreover relation (4.3.6) gives that

$$-\ln d(x,\partial\Omega) - C_1 \le 2\pi|\widetilde{\gamma}(x)| \le |\ln\varepsilon| + 2\pi$$

and therefore, provided $\varepsilon<1$,

$$d(x,\partial\Omega) \ge \varepsilon\exp\left(-2\pi - C_1\right) \equiv C_2\varepsilon.$$

Consequently $E\subset E_1$ with

$$E_1 = \{(x,y)\in\Omega\times\Omega, d(x,\partial\Omega)\ge C_2\varepsilon\}.$$

Moreover we have that $f'_\varepsilon(\gamma(x,y)) - f'_\varepsilon(G_{\mathbb{R}^2}(x,y)) = 0$ on the set

$$E' = \big\{(x,y)\in\Omega\times\Omega, |G_{\mathbb{R}^2}(x,y)| < \frac{1}{2\pi}|\ln\varepsilon| \text{ and } |\gamma(x,y)| < \frac{1}{2\pi}|\ln\varepsilon|\big\}.$$

Since $E\subset(E')^c$, assuming that $\varepsilon < \frac{1}{\operatorname{diam}\Omega}$ we have that $E\subset E_2\cup E_3$ with

$$E_2 = \{(x,y)\in\Omega\times\Omega, |x-y|\le\varepsilon\}$$

and

$$E_3 = \big\{(x,y)\in\Omega\times\Omega, |\gamma(x,y)| \ge \frac{1}{2\pi}|\ln\varepsilon|\big\}.$$

Using the fact that $E\subset(E_1\cap E_2)\cup(E_1\cap E_3)$ as well as relation (4.3.8) into relation (4.4.5) yields that

$$\begin{aligned}\iint_{\Omega\times\Omega} p(x,y)|A_{2,\varepsilon}(x,y)|\mathrm{d}x\mathrm{d}y \le \iint_{E_1\cap E_2} & p(x,y)B_\varepsilon(x,y)\mathrm{d}x\mathrm{d}y \\ & + \iint_{E_1\cap E_3} p(x,y)B_\varepsilon(x,y)\mathrm{d}x\mathrm{d}y \end{aligned} \tag{4.4.6}$$

with

$$B_\varepsilon(x,y) = 2|\nabla_x\gamma(x,y)|\frac{C}{d(x,\partial\Omega)}.$$

We bound now the quantity $\iint_{E_1\cap E_2} p(x,y)B_\varepsilon(x,y)\mathrm{d}x\mathrm{d}y$. Let $x\in\Omega$ be such that $d(x,\partial\Omega)\ge C_2\varepsilon$. We have that $\{y\in\Omega,(x,y)\in E_2\} = D(x,\varepsilon)\cap\Omega$. Since $\nabla_x\gamma$ is harmonic in both its variables, by the maximum principle we have that

$$\sup_{y\in D(x,\varepsilon)\cap\Omega}|\nabla_x\gamma(x,y)| \le \sup_{y\in\partial(D(x,\varepsilon)\cap\Omega)}|\nabla_x\gamma(x,y)|.$$

Since $\partial(D(x,\varepsilon)\cap\Omega)\subset\{y\in\mathbb{C},|x-y|=\varepsilon\}\cup\partial\Omega$, and $d(x,\partial\Omega)\geq C_2\varepsilon$, we have that for every $y\in\partial(D(x,\varepsilon)\cap\Omega)$, there exists a constant C independent of x and y such that $|x-y|>C\varepsilon$. Since $|\nabla_x\gamma(x,y)|\leq\frac{C}{x-y}$, for every $y\in\partial(D(x,\varepsilon)\cap\Omega)$ we have that $|\nabla_x\gamma(x,y)|\leq\frac{C}{\varepsilon}$. Therefore,

$$\sup_{y\in D(x,\varepsilon)\cap\Omega}|\nabla_x\gamma(x,y)|\leq\frac{C}{\varepsilon}$$

and thus in the case $p(x,y)=\frac{1}{d(x,\partial\Omega)^\kappa}$,

$$\begin{aligned}\iint_{E_1\cap E_2}p(x,y)B_\varepsilon(x,y)\mathrm{d}x\mathrm{d}y&\leq\frac{C}{\varepsilon}\iint_{E_1\cap E_2}\frac{1}{d(x,\partial\Omega)^{1+\kappa}}\mathrm{d}x\mathrm{d}y\\&\leq\frac{C}{\varepsilon}2\pi\varepsilon^2\int_{\{x,d(x,\partial\Omega)\geq C\varepsilon\}}\frac{1}{d(x,\partial\Omega)^{1+\kappa}}\mathrm{d}x\\&\leq C\varepsilon^{1-\kappa}\\&\leq C\end{aligned}$$

where we used Lemma 4.2.1.

For the other expression of $p(x,y)$ we directly use that $|\nabla_x\gamma(x,y)|\leq\frac{C}{|x-y|}$ as well as relation (4.3.8) to obtain that

$$\begin{aligned}\iint_{E_1\cap E_2}p(x,y)B_\varepsilon(x,y)\mathrm{d}x\mathrm{d}y&\leq\iint_{E_1\cap E_2}\frac{C}{|x-y|^{1+\kappa}}\frac{1}{d(x,\partial\Omega)}\mathrm{d}x\mathrm{d}y\\&\leq\int_{d(x,\partial\Omega)\geq C_2\varepsilon}\frac{C}{d(x,\partial\Omega)}\Big(\int_{|x-y|\leq\varepsilon}\frac{1}{|x-y|^{1+\kappa}}\mathrm{d}y\Big)\mathrm{d}x\\&\leq C\varepsilon^{1-\kappa}\int_{d(x,\partial\Omega)\geq C_2\varepsilon}\frac{C}{d(x,\partial\Omega)}\mathrm{d}x\\&\leq C\varepsilon^{1-\kappa}|\ln(\varepsilon)|\\&\leq C.\end{aligned}$$

Therefore

$$\iint_{E_1\cap E_2}p(x,y)B_\varepsilon(x,y)\mathrm{d}x\mathrm{d}y<C$$

for both choices of p.

Now we need to estimate the quantity $\iint_{E_1\cap E_3}p(x,y)B_\varepsilon(x,y)\mathrm{d}x\mathrm{d}y$. We start by recalling that since γ is smooth on $\Omega\times\Omega$ and symmetric, for ε small enough, the relation $(x,y)\in E_3$ implies that either $x\in V_j$ or $y\in V_j$ for an index j. By Lemma 4.3.3, we know that on $V_j\times\Omega$ the map $\gamma-\gamma_{\Omega_j}$ is bounded by a constant M. We conclude that $\forall(x,y)\in E_3$, $\frac{1}{2\pi}|\ln\varepsilon|-M\leq|\gamma_{\Omega_j}(x,y)|$ and therefore using Lemma 4.2.9 or Lemma 4.2.10 with $k=1$, we know that there exists a constant C such that

$$|x-y|\leq C\varepsilon.$$

So E_3 is included in a domain of similar form to E_2 and we can reproduce the previous argument and conclude that there exists a constant C independent of ε such that

$$\iint_{E_1\cap E_3}p(x,y)B_\varepsilon(x,y)\mathrm{d}x\mathrm{d}y<C.$$

Recalling relation (4.4.6), we have proved that

$$\iint_{\Omega\times\Omega}p(x,y)|A_{2,\varepsilon}(x,y)|\mathrm{d}x\mathrm{d}y<C$$

which concludes the proof of the lemma. □

Lemme 4.4.3. *There exists a constant C independent of ε such that for every $1 \leq j \leq m$,*

$$|\nabla\widetilde{\gamma}_\varepsilon(x) \cdot \beta_j(x)| \leq C. \tag{4.4.7}$$

Démonstration. Recalling that $|f_\varepsilon'| \leq 1$, this lemma is a direct consequence of the fact that $\nabla\widetilde{\gamma}_\varepsilon(x) = \nabla\widetilde{\gamma}(x) f_\varepsilon'(\widetilde{\gamma}(x))$ and of Corollary 4.3.6. □

4.4.2 End of the proof of Theorem 4.1.2

The end of the proof of Theorem 4.1.2 is largely inspired from the work previously done in [61].

Recall that $\Gamma = \{X = (x_1, \dots, x_N), d(X) > 0\}$ where

$$d(X) = \min\left(\min_{i\neq j}|x_i - x_j|, \min_i \mathrm{d}(x_i, \partial\Omega)\right) \quad \forall X = (x_1, \dots, x_N).$$

The aim of the Theorem 4.1.2 is to prove that $\tau(X) = \infty$ for λ-almost every X in Ω^N. Since $\lambda(\Omega^N \setminus \Gamma) = 0$, we can assume that $X \in \Gamma$. We denote by $S_t X = (x_1(t), \dots, x_N(t))$ the maximal solution of the point vortex system (4.1.5) with $(x_1(0), \dots, x_N(0)) = X$, and by $S_t^\varepsilon X = (x_1^\varepsilon(t), \dots, x_N^\varepsilon(t))$ the global solution of equations (4.4.1) which is the regularized dynamics with the same initial data X.

For any $X \in \Gamma$, we define $\tau_\varepsilon(X)$ as the supremum of all times such that the system of relations

$$\begin{cases} |G_{\mathbb{R}^2}(x_i(t), x_j(t))| < \frac{1}{2\pi}|\ln\varepsilon| \\ |\widetilde{\gamma}(x_i(t))| < \frac{1}{2\pi}|\ln\varepsilon| \\ |\gamma_\Omega(x_i(t), x_j(t))| < \frac{1}{2\pi}|\ln\varepsilon| \end{cases}$$

are true for any $i \neq j$ and any $t \in [0, \tau_\varepsilon(X)[$, with the convention that $\tau_\varepsilon(X) = 0$ if there exists no such t. By relations (4.4.2), we know that for every $X \in \Gamma$, $S_t^\varepsilon X = S_t X$ for every $t < \tau_\varepsilon(X)$.

Let us introduce the function

$$F(r) = \exp(-\eta r), \tag{4.4.8}$$

with a constant $0 < \eta < 1$ that we will specify later. Let $\phi_\varepsilon : \Gamma \to \mathbb{R}$ be defined by

$$\phi_\varepsilon(X) = \frac{1}{2}\sum_{i\neq j} F(G_\varepsilon(x_i, x_j)) + \frac{1}{2}\sum_i F(-\widetilde{\gamma}_\varepsilon(x_i)), \tag{4.4.9}$$

and $\Lambda_\varepsilon : \Gamma \times \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\Lambda_\varepsilon(X, t) = \frac{\mathrm{d}}{\mathrm{d}t}\phi_\varepsilon(S_t^\varepsilon X). \tag{4.4.10}$$

Since equations (4.4.1) are autonomous, we have that

$$\Lambda_\varepsilon(X, t) = \Lambda_\varepsilon(S_t^\varepsilon X, 0). \tag{4.4.11}$$

We now claim that the following proposition is true.

Proposition 4.4.4. *For every $X \in \Gamma$,*

$$\phi_\varepsilon(S^\varepsilon_{\tau_\varepsilon(X)}X) \geq \frac{1}{2}F\left(\frac{1}{8\pi}\ln(\varepsilon)\right) = \frac{1}{2}\varepsilon^{-\frac{\eta}{8\pi}}.$$

We delay the proof for the time being. Let $\tau > 0$ be a fixed time. By Proposition 4.4.4,

$$\{X \in \Gamma, \tau_\varepsilon(X) \leq \tau\} \subset \left\{X \in \Gamma, \sup_{t\in[0,\tau]} \phi(S_t^\varepsilon X) \geq \frac{1}{2}\varepsilon^{-\frac{\eta}{8\pi}}\right\},$$

Therefore,

$$\begin{aligned}\lambda(\{X \in \Gamma, \tau(X) \leq \tau\}) &\leq \lambda(\{X \in \Gamma, \tau_\varepsilon(X) \leq \tau\})\\ &\leq \lambda\left(\left\{X \in \Gamma, \sup_{t\in[0,\tau]} \phi_\varepsilon(S_t^\varepsilon X) \geq \frac{1}{2}\varepsilon^{-\frac{\eta}{8\pi}}\right\}\right)\\ &\leq 2\varepsilon^{\frac{\eta}{8\pi}} \int_\Gamma \sup_{t\in[0,\tau]} \phi_\varepsilon(S_t^\varepsilon X)\mathrm{d}\lambda(X).\end{aligned}$$

Recalling the definition (4.4.10) of Λ_ε and relation (4.4.11), for every $t \in [0, \tau]$ we have that

$$\phi_\varepsilon(S_t^\varepsilon X) = \phi_\varepsilon(X) + \int_0^t \Lambda_\varepsilon(X, s)\mathrm{d}s = \phi_\varepsilon(X) + \int_0^t \Lambda_\varepsilon(S_s^\varepsilon X, 0)\mathrm{d}s,$$

thus

$$\sup_{t\in[0,\tau]} \phi_\varepsilon(S_t^\varepsilon X) \leq |\phi_\varepsilon(X)| + \int_0^\tau |\Lambda_\varepsilon(S_s^\varepsilon X, 0)|\mathrm{d}s.$$

Using Fubini-Tonelli's Theorem we have that

$$\int_\Gamma \sup_{t\in[0,\tau]} \phi_\varepsilon(S_t^\varepsilon X)\mathrm{d}\lambda(X) \leq \int_\Gamma |\phi_\varepsilon(X)|\mathrm{d}\lambda(X) + \int_0^\tau \int_\Gamma |\Lambda_\varepsilon(S_s^\varepsilon X, 0)|\mathrm{d}\lambda(X)\mathrm{d}s.$$

Since the flow S^ε is Hamiltonian, it is area preserving (see [7, Corollary 1.10]) and thus we have that for any $s \in \mathbb{R}_+$,

$$\int_\Omega |\Lambda_\varepsilon(S_s^\varepsilon X, 0)|\mathrm{d}\lambda(X) = \int_\Omega |\Lambda_\varepsilon(X, 0)|\mathrm{d}\lambda(X).$$

We will prove later that there exists a constant A_0 depending only on Ω, N, η and on the masses $(a_i)_i$, such that for every $\varepsilon > 0$ and $t \in \mathbb{R}$, we have that

$$\int_\Gamma \phi_\varepsilon(X)\mathrm{d}\lambda(X) \leq A_0, \tag{4.4.12}$$

and

$$\int_\Gamma |\Lambda_\varepsilon(X, 0)|\, \mathrm{d}\lambda(X) \leq A_0. \tag{4.4.13}$$

Relations (4.4.12) and (4.4.13) yield that

$$\int_\Gamma \sup_{t\in[0,\tau]} \phi_\varepsilon(S_t^\varepsilon(X))\mathrm{d}\lambda(X) \leq A_0(1 + \tau)$$

and therefore

$$\lambda(\{X \in \Gamma, \tau(X) \leq \tau\}) \leq 2\varepsilon^{\frac{\eta}{8\pi}} A_0(1 + \tau).$$

This being true for every $\varepsilon > 0$, and given that the left-hand side of the equation doesn't depend on ε, letting $\varepsilon \to 0$ yields

$$\lambda(\{X \in \Gamma, \tau(X) \leq \tau\}) = 0.$$

This is true for every time $\tau > 0$, and since

$$\{X \in \Gamma, \tau(X) < \infty\} = \bigcup_{k\in\mathbb{N}^*}^{\infty} \{X \in \Gamma, \tau(X) < k\},$$

we have the desired result :

$$\lambda\{X \in \Gamma, \tau(X) < \infty\} = 0.$$

Proof of Proposition 4.4.4

We recall that for every $t \leq \tau_\varepsilon(X)$ and any $i \neq j$, we have that $G_\varepsilon(x_i, x_j) = G(x_i, x_j)$, $\gamma_\varepsilon(x_i, x_j) = \gamma(x_i, x_j)$ and $\widetilde{\gamma}_\varepsilon(x_i) = \widetilde{\gamma}(x_i)$. Let $(x, y) \in \Omega$, $x \neq y$. We recall that at the time $t = \tau_\varepsilon(X)$, there exist $i \neq j$ such that either

$$|G_{\mathbb{R}^2}(x_i^\varepsilon(\tau_\varepsilon(X)), x_j^\varepsilon(\tau_\varepsilon(X)))| = \frac{1}{2\pi}|\ln \varepsilon|$$

or

$$|\gamma_\Omega(x_i^\varepsilon(\tau_\varepsilon(X)), x_j^\varepsilon(\tau_\varepsilon(X)))| = \frac{1}{2\pi}|\ln \varepsilon|$$

or

$$|\widetilde{\gamma}(x_i^\varepsilon(\tau_\varepsilon(X)))| = \frac{1}{2\pi}|\ln \varepsilon|.$$

Recalling the definition of Φ given by relation (4.4.9), and the fact that F is positive, we have that

$$\phi_\varepsilon(S^\varepsilon_{\tau_\varepsilon(X)}X) \geq \max\left(\frac{1}{2}F(G_\varepsilon(x_i^\varepsilon(\tau_\varepsilon(X)), x_j^\varepsilon(\tau_\varepsilon(X))), \frac{1}{2}F(-\widetilde{\gamma}_\varepsilon(x_i^\varepsilon(\tau_\varepsilon(X))))\right).$$

Therefore, since F is decreasing, in order to prove Proposition 4.4.4 it is enough to prove the following lemma.

Lemme 4.4.5. *Let $(x, y) \in \Omega \times \Omega$. If one of the conditions*

$$\begin{cases} |G_{\mathbb{R}^2}(x, y)| \geq \frac{1}{2\pi}|\ln \varepsilon| \\ |\gamma(x, y)| \geq \frac{1}{2\pi}|\ln \varepsilon| \\ |\widetilde{\gamma}(x)| \geq \frac{1}{2\pi}|\ln \varepsilon| \end{cases}$$

is true, then it implies that either

$$G(x, y) \leq \frac{1}{8\pi}\ln \varepsilon$$

or

$$\widetilde{\gamma}(x) \geq -\frac{1}{8\pi}\ln \varepsilon.$$

Démonstration. Firstly, provided that ε is small enough such that $-\frac{1}{2\pi}|\ln \varepsilon| < \min_\Omega \widetilde{\gamma}$, which is possible by relation (4.3.7), the relation $|\widetilde{\gamma}(x)| \geq \frac{1}{2\pi}|\ln \varepsilon|$ implies that $\widetilde{\gamma}(x) \geq -\frac{1}{2\pi}\ln \varepsilon \geq -\frac{1}{8\pi}\ln \varepsilon$.

Secondly assume that $|\gamma(x, y)| \geq \frac{1}{2\pi}|\ln \varepsilon|$. We recall the decomposition (4.3.1). Since K is a compact set, the map γ is bounded on $K \times K$. Therefore, provided ε is small enough such that $\frac{1}{2\pi}|\ln \varepsilon| > \max_{K\times K}|\gamma|$, we have that the condition $|\gamma(x, y)| \geq \frac{1}{2\pi}|\ln \varepsilon|$ implies that there exists $0 \leq j \leq m$ such that either $x \in V_j$ or $y \in V_j$. By symmetry, we assume that $x \in V_j$.

We recall Lemma 4.3.3 which states that the map $\gamma - \gamma_{\Omega_j}$ is bounded on $V_j \times \Omega$. Therefore, there exists a constant $M > 0$ such that $|\gamma_{\Omega_j}(x, y)| \geq \frac{1}{2\pi}|\ln \varepsilon| - M$. If $j = 0$, we use Lemma 4.2.9 with $k = 1$ and $U = \Omega_j$, else we use Lemma 4.2.10 with $k = 1$ and $\Pi = \Omega_j$, and $\mathcal{U} = \Omega$, to obtain that $d(x, \partial\Omega) \leq C\varepsilon$. Therefore relation (4.3.6) gives that

$$2\pi\widetilde{\gamma}(x) \geq -\ln(C\varepsilon) - C_1.$$

We deduce that there exists $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ we have

$$\widetilde{\gamma}(x) \geq -\frac{1}{8\pi}\ln(\varepsilon).$$

Thirdly, assume that $|G_{\mathbb{R}^2}(x, y)| \geq \frac{1}{2\pi}|\ln \varepsilon|$, which is equivalent to $G_{\mathbb{R}^2}(x, y) \leq \frac{1}{2\pi}\ln \varepsilon$ provided that ε is smaller than $\frac{1}{\operatorname{diam}\Omega}$. Recalling relation (4.1.3), then either $G(x, y) \leq \frac{1}{4\pi}\ln \varepsilon$ or $-\gamma(x, y) \leq \frac{1}{4\pi}\ln \varepsilon$. The condition $G(x, y) \leq \frac{1}{4\pi}\ln \varepsilon$ naturally implies that $G(x, y) \leq \frac{1}{8\pi}\ln \varepsilon$. Assume now that $\gamma(x, y) \geq -\frac{1}{4\pi}\ln \varepsilon$. As in the second case, we use Lemma 4.2.9 or Lemma 4.2.10 with $k = \frac{1}{2}$ to obtain that $d(x, \partial\Omega) \leq C\sqrt{\varepsilon}$, which leads by relation (4.3.6) to

$$\widetilde{\gamma}(x) \geq -\frac{1}{8\pi}\ln(\varepsilon),$$

for ε small enough. The lemma is now proved, which concludes the proof of Proposition 4.4.4. □

Proof of relations (4.4.12) **and** (4.4.13)**.**

We start by proving (4.4.12). Recalling the definitions of ϕ_ε and F given by relations (4.4.9) and (4.4.8) we have that

$$\phi_\varepsilon(X) = \frac{1}{2}\sum_{i\neq j} \exp(-\eta G_\varepsilon(x_i,x_j)) + \frac{1}{2}\sum_i \exp(\eta\widetilde{\gamma}_\varepsilon(x_i)).$$

Since $\eta > 0$ and exp is an increasing function, relations (4.4.3) yield that

$$|\phi_\varepsilon(X)| \le \sum_{i\neq j} \exp(\eta|G(x_i,x_j)|) + \sum_i \exp(\eta|\widetilde{\gamma}(x_i)|).$$

Relation (4.3.6) gives that $|\widetilde{\gamma}(x_i)| \le -\frac{1}{2\pi}\ln d(x_i,\partial\Omega) + C_3$. Using also (4.3.3), we have that

$$|\phi_\varepsilon(X)| \le \frac{1}{2}\sum_{i\neq j} \exp(\eta C(1+|\ln|x_i-x_j||)) + \frac{1}{2}\sum_i \exp(-\frac{\eta}{2\pi}\ln d(x_i,\partial\Omega) + \eta C_3).$$

Since $|\ln|x-y|| \le \max\{-\ln|x-y|, \ln(\operatorname{diam}\Omega)\}$, we bound $|\ln|x-y|| \le -\ln|x-y| + C$ and thus

$$|\phi_\varepsilon(X)| \le \exp(\eta C_4)\left[\frac{1}{2}\sum_{i\neq j}\left(\frac{1}{|x_i-x_j|^{\eta C}}\right) + \frac{1}{2}\sum_i \frac{1}{d(x_i,\partial\Omega)^{\eta/2\pi}}\right].$$

Choosing $\eta < \min\{\frac{2}{C}, 2\pi\}$, and noticing that the last expression doesn't depend on ε, we obtain that $\int_\Gamma \phi_\varepsilon(X)\mathrm{d}\lambda(X)$ is bounded independently of ε. This proves relation (4.4.12).

We now want to prove relation (4.4.13). By the definition of Λ_ε given in relation (4.4.10), we have that

$$\int_\Gamma |\Lambda_\varepsilon(X,0)|\,\mathrm{d}\lambda(X) = \int_\Gamma \left|\frac{\mathrm{d}}{\mathrm{d}t}[\phi_\varepsilon(S_t^\varepsilon X)]\big|_{t=0}\right| \mathrm{d}\lambda(X).$$

Therefore in order to prove relation (4.4.13) we have to show that at time $t=0$, the quantity $\frac{\mathrm{d}}{\mathrm{d}t}\phi_\varepsilon(S_t^\varepsilon X)$ is bounded in $L^1(\Gamma)$ independently of ε. Let us compute :

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\phi_\varepsilon(S_t^\varepsilon X) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\sum_{i\neq j} F(G_\varepsilon(x_i^\varepsilon(t),x_j^\varepsilon(t))) + \frac{1}{2}\sum_i F(-\widetilde{\gamma}_\varepsilon(x_i^\varepsilon(t)))\right] \\
&= \frac{1}{2}\sum_{i\neq j} F'(G_\varepsilon(x_i^\varepsilon(t),x_j^\varepsilon(t)))\nabla_x G_\varepsilon(x_i^\varepsilon(t),x_j^\varepsilon(t))\cdot\frac{\mathrm{d}x_i^\varepsilon(t)}{\mathrm{d}t} \\
&\quad + \frac{1}{2}\sum_{i\neq j} F'(G_\varepsilon(x_i^\varepsilon(t),x_j^\varepsilon(t)))\nabla_y G_\varepsilon(x_i^\varepsilon(t),x_j^\varepsilon(t))\cdot\frac{\mathrm{d}x_j^\varepsilon(t)}{\mathrm{d}t} \\
&\quad - \frac{1}{2}\sum_i F'(-\widetilde{\gamma}_\varepsilon(x_i(t)))\nabla\widetilde{\gamma}_\varepsilon(x_i^\varepsilon(t))\cdot\frac{\mathrm{d}x_i^\varepsilon(t)}{\mathrm{d}t}.
\end{aligned}$$

However, one can notice that since $G_\varepsilon(x,y) = G_\varepsilon(y,x)$ we have that $\nabla_y G_\varepsilon(x,y) = \nabla_x G_\varepsilon(y,x)$ and thus

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\phi_\varepsilon(S_t^\varepsilon X) &= \sum_{i\neq j} F'(G_\varepsilon(x_i^\varepsilon(t),x_j^\varepsilon(t)))\nabla_x G_\varepsilon(x_i^\varepsilon(t),x_j^\varepsilon(t))\cdot\frac{\mathrm{d}x_i^\varepsilon(t)}{\mathrm{d}t} \\
&\quad - \frac{1}{2}\sum_i F'(-\widetilde{\gamma}_\varepsilon(x_i^\varepsilon(t)))\nabla\widetilde{\gamma}_\varepsilon(x_i^\varepsilon(t))\cdot\frac{\mathrm{d}x_i^\varepsilon(t)}{\mathrm{d}t}.
\end{aligned}$$

We recall relation (4.4.1) :

$$\frac{\mathrm{d}x_i^\varepsilon(t)}{\mathrm{d}t} = \sum_{\substack{k=1\\k\neq i}}^N \nabla_x^\perp G_\varepsilon(x_i^\varepsilon(t), x_k^\varepsilon(t))a_k + \frac{1}{2}\nabla^\perp\widetilde{\gamma}_\varepsilon(x_i^\varepsilon(t))a_i + \sum_{k=1}^m c_k(t)\beta_k(x_i^\varepsilon(t)).$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\phi_\varepsilon(S_t^\varepsilon X) &= \sum_{\substack{i\neq j\\k\neq i}} F'(G_\varepsilon(x_i^\varepsilon(t), x_j^\varepsilon(t)))\nabla_x G_\varepsilon(x_i^\varepsilon(t), x_j^\varepsilon(t))\cdot\nabla_x^\perp G_\varepsilon(x_i^\varepsilon(t), x_k(t))a_k\\
&+\sum_{i\neq j} F'(G_\varepsilon(x_i^\varepsilon(t), x_j^\varepsilon(t)))\nabla_x G_\varepsilon(x_i^\varepsilon(t), x_j^\varepsilon(t))\cdot\frac{1}{2}\nabla^\perp\widetilde{\gamma}_\varepsilon(x_i^\varepsilon(t))a_i\\
&+\sum_{i\neq j} F'(G_\varepsilon(x_i^\varepsilon(t), x_j^\varepsilon(t)))\nabla_x G_\varepsilon(x_i^\varepsilon(t), x_j^\varepsilon(t))\cdot\sum_{k=1}^m c_k(t)\beta_k(x_i^\varepsilon(t))\\
&-\frac{1}{2}\sum_i F'(-\widetilde{\gamma}_\varepsilon(x_i^\varepsilon(t)))\nabla\widetilde{\gamma}_\varepsilon(x_i^\varepsilon(t))\cdot\sum_{\substack{k=1\\k\neq i}}^N\nabla_x^\perp G_\varepsilon(x_i^\varepsilon(t), x_k^\varepsilon(t))a_k\\
&-\frac{1}{4}\sum_i F'(-\widetilde{\gamma}_\varepsilon(x_i^\varepsilon(t)))\nabla\widetilde{\gamma}_\varepsilon(x_i^\varepsilon(t))\cdot\nabla^\perp\widetilde{\gamma}_\varepsilon(x_i^\varepsilon(t))a_i\\
&-\frac{1}{2}\sum_i F'(-\widetilde{\gamma}_\varepsilon(x_i^\varepsilon(t)))\nabla\widetilde{\gamma}_\varepsilon(x_i^\varepsilon(t))\cdot\sum_{k=1}^m c_k(t)\beta_k(x_i^\varepsilon(t))\\
&\equiv B_1(t)+B_2(t)+B_3(t)+B_4(t)+B_5(t)+B_6(t).\end{aligned}$$

We recall that $x_i^\varepsilon(0) = x_i$, where $X = (x_1, \dots, x_N)$. First of all, we observe that $B_5(t) = 0$. Notice that from relations (4.4.3) we have that

$$|\nabla_x G_\varepsilon(x, y)| \leq \frac{C}{|x-y|} \tag{4.4.14}$$

where the constant C is independent of ε.

The same estimates as at the beginning of the proof of relations (4.4.12) and (4.4.13) show that for any $X \in \Gamma$ we have that

$$|F'(G_\varepsilon(x_i, x_j))| \leq \frac{C'}{|x_i - x_j|^{\eta C}} \tag{4.4.15}$$

and

$$|F'(-\widetilde{\gamma}_\varepsilon(x_i))| \leq \frac{C}{d(x_i, \partial\Omega)^{\eta/2\pi}} \tag{4.4.16}$$

where we used that $0 < \eta < 1$. Relation (4.4.16), together with relation (4.4.7), yields

$$|B_6(0)| \leq \sum_i \frac{C}{d(x_i, \partial\Omega)^{\eta/2\pi}}.$$

We also have that

$$|c_k(0)\beta_k(x_i)| \leq C,$$

and therefore using relations (4.4.15) and (4.4.14) we have that

$$|B_3(0)| \leq \sum_{i\neq j}\frac{C'}{|x_i - x_j|^{\eta C+1}}.$$

Both $B_3(0)$ and $B_6(0)$ are therefore bounded in $L^1(\Gamma)$ uniformly in ε provided that η is small enough.

Using relation (4.4.15), we have that

$$|B_2(0)| \leq \sum_{i\neq j} \frac{C'}{|x_i - x_j|^{\eta C}} |\nabla_x G_\varepsilon(x_i, x_j) \cdot \nabla^\perp \widetilde{\gamma}_\varepsilon(x_i)|$$

and thus Lemma 4.4.2 implies that $B_2(0)$ is bounded in $L^1(\Gamma)$ uniformly in ε if η is small enough. Similarly, using this time relation (4.4.16), we have that

$$|B_4(0)| \leq \sum_{i\neq k} \frac{C}{d(x_i, \partial\Omega)^{\eta/2\pi}} |\nabla \widetilde{\gamma}_\varepsilon(x_i) \cdot \nabla_x^\perp G_\varepsilon(x_i, x_k)|$$

and once again, Lemma 4.4.2 allows to conclude that $B_4(0)$ is bounded in $L^1(\Gamma)$ uniformly in ε if η is small enough.

Finally, we bound $B_1(0)$ by noticing first that for $k = j$, the expression $\nabla_x G_\varepsilon(x_i^\varepsilon(t), x_j^\varepsilon(t)) \cdot \nabla_x^\perp G_\varepsilon(x_i^\varepsilon(t), x_k(t))$ vanishes, and therefore using relation (4.4.15) and the relation (4.4.14) we obtain that

$$|B_1(0)| \leq \sum_{\substack{i\neq j\\ k\neq i\\ k\neq j}} \frac{C'}{|x_i - x_k||x_i - x_j|^{\eta C+1}}.$$

This term is bounded in $L^1(\Gamma)$ uniformly in ε when η is small enough. This concludes the proof of relation (4.4.13). The proof of Theorem 4.1.2 is completed.

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Chapitre 5

Hölder regularity of the trajectories

This chapter is constituted of the following preprint :

Hölder regularity of collapses of point vortices
With **Ludovic Godard-Cadillac**, arXiv :2111.14230.

The first part of this article studies the collapses of point-vortices for the Euler equation in the plane and for surface quasi-geostrophic equations in the general setting of α models. In these models the kernel of the Biot-Savart law is a power function of exponent $-\alpha$. It is proved that, under a standard non-degeneracy hypothesis, the trajectories of the point-vortices have a Hölder regularity up to, and including, the time of collapse. The Hölder exponent obtained is $1/(\alpha+1)$ and this exponent is proved to be optimal for all α by exhibiting an example of a 3-vortex collapse.

The same question is then addressed for the Euler point-vortex system in smooth bounded connected domains. It is proved that if a given point-vortex has an accumulation point in the interior of the domain as $t \to T$, then it converges towards this point and displays the same Hölder continuity property. A partial result for point-vortices that collapse with the boundary is also established : we prove that their distance to the boundary is Hölder regular.

Introduction

The point-vortex model is a system of ordinary differential equations introduced by Helmholtz [43] that is a standard model for uniform inviscid planar fluid mechanics. This system is derived from the two-dimensional Euler equations written in terms of vorticity. Typically, it gives account of the natural situation where the vorticity of the fluid is sharply concentrated around some points $x_i \in \mathbb{R}^2$ with $i = 1 \dots N$ and is (formally) replaced by Dirac masses. These points then evolve in the plane according to the Euler equations. This system of differential equations has been widely studied in its different aspects, both for itself and for its links with PDE in fluid mechanics. For an introduction to the Euler point-vortex system, we refer to [62, chap. 4]. For a more extensive presentation, see [65]. This point-vortex model is used to describe vortex phenomena arising in different problems of fluid mechanics (see for instance : [31, 38, 53, 55, 61] and references therein). The question of a rigorous derivation of the point-vortex system from the Euler equations (also called desingularization problem, or localization problem) is a standard problem that is linked to the problem of confinement and localization of vorticity [22, 62, 74].

The present article focuses on the study of collapses of point-vortices at finite time $T > 0$. The existence of collapses for the Euler point-vortex problem in the plane has been obtained independently by [2, 39, 67]. The existence of collapses of point-vortices for all bounded smooth domains has been obtained recently by [37]. In [61], the authors prove that when the domain is a disc, the collapses are improbable in the sense that the set of initial datum leading to a collapse in finite time has a vanishing Lebesgue measure. This result has been extended to any bounded smooth domains (simply connected and multi-connected domains) by [20]. In [62, chap. 4], the authors obtained the improbability of collapses when the domain is the whole plane, under the non-neutral clusters hypothesis (5.1.19) presented hereafter. This hypothesis has been weakened by [34]. In that last article, the question of the behaviors of the point-vortices during a collapse is addressed : it is proved in the case of Euler equation in the plane that, under the non-neutral clusters hypothesis, the positions of the point-vortices converge as t goes to the time of collapse.

In the context of geophysical fluids, a standard model is given by the surface quasi-geostrophic equations. These equations have many common properties with the Euler equations. An extension of the point-vortex theory to these equations has been proposed recently by [29]. In that article, the authors study the desingularization problem in order to obtain a rigorous derivation of the system (see also [18, 73, 35]) and the improbability of collapses of point-vortices. This result was improved in [34]. The collapses for the quasi-geostrophic point-vortices are studied in [10, 33, 72] for the 3 vortex problem.

In [33], the second author of the present article studied the collapses for the 3 vortex problem in the Euler and in the quasi-geostrophic case. It is proved, under the non-neutral clusters hypothesis, that in the presence of a collapse, the trajectories are Hölder continuous, with an exponent that depends explicitly on the singularity of the vorticity kernel. Nevertheless, the proof in the previous paper fails to extend to the case of N vortices.

In the present article, we deal with this problem but the approach is different and we prove that the conjectured Hölder estimate does hold with N vortices. More precisely, we prove in Theorem 5.1.3 hereafter that for general α-models of point-vortex dynamics, under the non neutral clusters hypothesis, the trajectories of any solution well-defined on $[0, T)$ can be extended as a $\frac{1}{1+\alpha}$-Hölder regular trajectory on $[0, T]$. We also study the degenerate case where the total sum of the intensities is vanishing and we obtain similar estimates for the relative dynamics.

We also study the case of the Euler point-vortex dynamics in the case of a smooth bounded domain Ω. With the same assumptions and similar arguments, we prove in Theorem 5.1.4 that the trajectories of the point-vortices that remain far from the boundary are $\frac{1}{2}$-Hölder regular. We also prove for the other points that their distance to the boundary is $\frac{1}{2}$-Hölder regular, and goes to 0 at the time of collision.

Finally, in the appendix, we prove the existence of self-similar collapses with 3 vortices for a large class of point-vortex systems. This study gives in particular the optimality of the obtained Hölder exponent.

Our method of proof for the two theorems relies on a *clusterization* by induction. We group point-vortices into clusters and show that our construction has a finite number of steps and stays stable in a large enough interval of time. The estimate on the interval of time relies on a precise study of the dynamics for the centers of vorticity of the clusters. It is proved that point-vortices that are far from each other remain far from each other on a such interval of time.

We note that the same problem has already been addressed for the classical N-body gravitational problem [70] for the study of collisions.

5.1 Presentation of the problem and main results

5.1.1 Presentation of the point-vortex equations

Point-vortex for the Euler equations

The motion of an homogeneous inviscid fluid in the plane is given by the two-dimensional Euler equations. Using the vorticity formulation, these equations take the form

$$\begin{cases} \dfrac{\partial \omega}{\partial t} + v \cdot \nabla \omega = 0, \\ v = -\nabla^\perp (-\Delta)^{-1}\, \omega. \end{cases} \tag{Eu}$$

In the equations above, $v : [0,T]\times\mathbb{R}^2 \to \mathbb{R}^2$ is the velocity field of the fluid and $\omega : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ is the vorticity. The vorticity can be deduced from the velocity with the formula $\omega = \partial_1 v_2 - \partial_2 v_1 = \nabla^\perp \cdot v$, where the notation $x^\perp := (-x_2, x_1)$ denotes the counter-clockwise rotation of angle $\pi/2$ in the plane. The first equation in (Eu) is a transport equation of the vorticity by the flow of the fluid which is assumed to be incompressible. The second equation is called the *Biot-Savart law.*

To obtain the point-vortex model, we assume that the vorticity at time $t = 0$ is a sum of Dirac masses $\sum_{i=1}^N a_i\, \delta_{x_i}$, where a_i is the intensity of the i^{th} vortex located at position x_i. The Euler equation (Eu) implies that for future times the i^{th}-vortex is still a Dirac mass with the same intensity a_i and with a position $x_i(t)$ that evolves according to the following equations :

$$\frac{\mathrm{d}}{\mathrm{d}t} x_i(t) := \frac{1}{2\pi} \sum_{\substack{j=1 \\ j\neq i}}^{N} a_j \frac{\big(x_i(t) - x_j(t)\big)^\perp}{\big|x_i(t) - x_j(t)\big|^2}. \tag{5.1.1}$$

To obtain these equations of evolution from the Euler equations (Eu), we compute the velocity of the fluid in the case where the vorticity is a sum of Dirac masses $\sum_{i=1}^N a_i\, \delta_{x_i(t)}$. This gives

$$v(t,x) = \frac{1}{2\pi} \sum_{j=1}^{N} a_j \frac{\big(x - x_j(t)\big)^\perp}{\big|x - x_j(t)\big|^2},$$

where we used that the Green function of the $(-\Delta)$ operator in the plane is given by :

$$x \longmapsto \frac{1}{2\pi} \ln\left(\frac{1}{|x|}\right).$$

Under the natural assumption that a given point-vortex does not interact with itself, the transport equation for the vorticity gives (5.1.1). For the sake of generalization, we define the function $G_\alpha : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$, with $\alpha \geq 0$ by

$$\begin{cases} G_1(x) = \ln(|x|) \\ G_\alpha(x) = \dfrac{|x|^{1-\alpha}}{1-\alpha} \quad \text{if } \alpha \neq 1. \end{cases} \tag{5.1.2}$$

In particular, it satisfies for every $\alpha \geq 0$ that

$$|\nabla G_\alpha(x)| = \frac{1}{|x|^\alpha}. \tag{5.1.3}$$

The function G_α for a fixed value of $\alpha \geq 0$ will be referred throughout this article as *the kernel profiles of exponent* α. The Euler point-vortex equation can then be written using the kernel profile G_α with $\alpha = 1$:

$$\frac{\mathrm{d}}{\mathrm{d}t} x_i(t) := \frac{1}{2\pi} \sum_{\substack{j=1 \\ j\neq i}}^{N} a_j \nabla^\perp G_1\big(x_i(t) - x_j(t)\big). \tag{5.1.4}$$

In the case of the Euler equations in a bounded simply-connected smooth connected domain of $\mathbb{R}^2$ with impermeability condition at the boundary, it is possible to proceed to an analogous construction (see [20]) and get

$$\frac{\mathrm{d}}{\mathrm{d}t}x_i(t) := a_i \nabla_x^\perp \gamma_\Omega\big(x_i(t), x_i(t)\big) - \sum_{\substack{j=1 \\ j\neq i}}^{N} a_j \nabla_x^\perp \mathcal{G}_\Omega\big(x_i(t), x_j(t)\big),$$

where $\mathcal{G}_\Omega$ is the Green function of the $(-\Delta)$ operator in Ω and where

$$\gamma_\Omega(x,y) := -\mathcal{G}_\Omega(x,y) - \frac{1}{2\pi}G_1\big(x-y\big) \tag{5.1.5}$$

is harmonic in both variables. This function γ_Ω satisfies also the following estimate :

$$\forall\, x, y \in \Omega, \quad \big|\nabla_x \gamma_\Omega(x,y)\big| \leq \frac{C_\Omega}{\operatorname{dist}\big(x, \partial\Omega\big)} \tag{5.1.6}$$

where C_Ω is a constant that depends only on Ω and where the distance of a point x to a set A is defined by

$$\operatorname{dist}(x, A) := \inf_{y\in A} |x-y|.$$

Indeed, we know (see for instance [20]) that γ_Ω satisfies

$$\forall\, x, y \in \Omega, \quad \big|\nabla_x \gamma_\Omega(x,y)\big| \leq \frac{C_\Omega}{|x-y|} \tag{5.1.7}$$

and since the map $y \mapsto \nabla_x \gamma_\Omega(x,y)$ is harmonic, then by the maximum principle

$$|\nabla_x \gamma_\Omega(x,y)| \leq \max_{z\in\partial\Omega} |\nabla_x \gamma_\Omega(x,z)| \leq \max_{z\in\partial\Omega} \frac{C_\Omega}{|x-z|} = \frac{C_\Omega}{\operatorname{dist}(x,\partial\Omega)}.$$

In the case where the domain is non simply connected, an additional contribution must be taken into account coming from the circulation of the fluid around the holes of the domain. The M holes in the domain Ω are noted Ω_m and the domain delimited by the exterior boundary, namely Ω "without holes" is denoted by Ω_0. For any of these holes, the total circulation of the flow at the surface $\partial\Omega_m$ is preserved by the motion and is denoted by ξ_m. The equations for the point-vortex system in a non simply connected domain is given by (see for instance the Biot-Savart law given by [48, Proposition 2.9]) :

$$\frac{\mathrm{d}}{\mathrm{d}t}x_i(t) := a_i \nabla_x^\perp \gamma_\Omega\big(x_i(t), x_i(t)\big) - \sum_{\substack{j=1 \\ j\neq i}}^{N} a_j \nabla_x^\perp \mathcal{G}_\Omega\big(x_i(t), x_j(t)\big) + \sum_{m=1}^{M} c_m(t)\beta_m\big(x_i(t)\big). \tag{5.1.8}$$

In the equations above, the functions β_m for $m \in \{1, \dots, M\}$ are the harmonic vector fields, defined by

$$\begin{cases} \nabla\cdot\beta_m = 0 & \text{in } \Omega \\ \operatorname{curl}\beta_m = 0 & \text{in } \Omega \\ \beta_m \cdot n = 0 & \text{on } \partial\Omega \\ \displaystyle\int_{\Gamma_\ell} \beta_m \cdot (-n^\perp)\mathrm{d}s = \delta_{m,\ell} & \forall \ell \in \{1,\dots,M\}, \end{cases} \tag{5.1.9}$$

where $\delta_{m,\ell}$ represents here the Kronecker symbol and n is the exterior normal vector. The quantity c_m is given by :

$$c_m(t) = \xi_m + \sum_{k=1}^{N} a_k\, w_m\big(x_k(t)\big), \tag{5.1.10}$$

where the w_m are the harmonic maps defined by

$$\begin{cases} -\Delta w_m = 0 & \text{in } \Omega, \\ w_m = 1 & \text{on } \partial\Omega_m, \\ w_m = 0 & \text{on } \partial\Omega \setminus \partial\Omega_m. \end{cases}$$

Note that the extra term appearing in the case of non simply connected domains is a bounded term, by standard elliptic estimates since $\partial\Omega$ is smooth. Using the relation (5.1.5), the evolution equation (5.1.8) can be rewritten under the following expanded form :

$$\frac{\mathrm{d}}{\mathrm{d}t}x_i(t) = \sum_{j=1}^{N} a_j \nabla_x^{\perp}\gamma_\Omega\big(x_i(t), x_j(t)\big) + \frac{1}{2\pi}\sum_{\substack{j=1\\ j\neq i}}^{N} a_j \nabla^{\perp} G_1\big(x_i(t) - x_j(t)\big) + \sum_{m=1}^{M} c_m(t)\beta_m\big(x_i(t)\big). \tag{5.1.11}$$

For a wider introduction to the Euler point-vortex system in bounded domains, we refer to [61, chap. 4] or to [65]. For more details on the Green's function in bounded domains, see [40].

Point-vortex for the quasi-geostrophic equations

In geophysical fluid dynamics, a standard model is given by the surface quasi-geostrophic equations :

$$\begin{cases} \dfrac{\mathrm{d}\omega}{\mathrm{d}t} + v\cdot\nabla\omega = 0, \\ v = -\nabla^{\perp}(-\Delta)^{-s}\,\omega, \end{cases} \tag{SQG}$$

with $0 < s < 1$. These equations give account of a quasi-stratified fluid subject to Brünt-Väisälä oscillations evolving in a rapidly rotating frame. This model is widely used for weather forecast [69, 79]. The transport equation of the vorticity in (SQG) is the same as in (Eu). The difference lays in the Biot-Savart law that involves a fractional Laplace operator in this case. Formally, if we take $s = 1$ in (SQG) then it gives back (Eu). In the plane, the Green function of the fractional Laplace operator is given by

$$x \longmapsto \frac{C_s}{|x|^{2(1-s)}}, \qquad \text{where} \qquad C_s := \frac{\Gamma(1-s)}{2^{2s}\,\pi\,\Gamma(s)},$$

with Γ the standard Gamma function.

In [29], the authors suggested to exploit the proximity of formulations between (Eu) and (SQG) to derive a more general point-vortex model. Indeed we formally choose an initial value for ω in (SQG) to be a sum of Dirac masses $\sum_{i=1}^N a_i\,\delta_{x_i}$. A reasoning similar to the case of the Euler equation (but involving this time the fractional Green function) gives that the structure of Dirac masses persists and the position of the Dirac masses evolves in time according to

$$\frac{\mathrm{d}}{\mathrm{d}t}x_i(t) = 2C_s(1-s)\sum_{\substack{j=1\\ j\neq i}}^{N} a_j \frac{\big(x_i(t) - x_j(t)\big)^{\perp}}{\big|x_i(t) - x_j(t)\big|^{4-2s}}. \tag{5.1.12}$$

Similarly to (5.1.4), it is possible to reformulate this system (5.1.12) using the kernel profiles G_α defined by (5.1.2) with $1 < \alpha < 3$. More precisely, the quasi-geostrophic point-vortex model (5.1.12) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}x_i(t) = 2C_s(1-s)\sum_{\substack{j=1\\ j\neq i}}^{N} a_j \nabla^{\perp} G_{(3-2s)}\big(x_i(t) - x_j(t)\big). \tag{5.1.13}$$

These reformulations involving the kernel profiles G_α allows us to consider the general α-point-vortex model in the plane. It is defined for all $\alpha \geq 0$ by the following differential equation :

$$\frac{\mathrm{d}}{\mathrm{d}t}x_i(t) := \sum_{\substack{j=1\\ j\neq i}}^{N} a_j \, \nabla^\perp G_\alpha\big(x_i(t) - x_j(t)\big) = \sum_{\substack{j=1\\ j\neq i}}^{N} a_j \, \frac{(x_i(t) - x_j(t))^\perp}{|x_i(t) - x_j(t)|^{\alpha+1}}. \tag{5.1.14}$$

Up to a time rescaling, the point-vortex model for the Euler equations in the plane (5.1.4) or quasi-geostrophic point-vortex model (5.1.13) are particular cases of the more general evolution system (5.1.14) that we study in this article.

The point-vortex model (5.1.14) admits a unique solution by the Cauchy-Lipschitz theorem provided that the right-hand side of (5.1.14) remains bounded and Lipschitz. This means that this system is well posed in the absence of collapses of point-vortices. In other words, if $T > 0$ the maximal time of existence of the solution is finite then

$$\liminf_{t\to T^-} \min_{i\neq j} |x_i(t) - x_j(t)| = 0. \tag{5.1.15}$$

The study of collapse situations (5.1.15) has two main interests. First, understanding collapses permits to avoid them and ensures the system to be well-defined for all times. The other advantage is that understanding the collapses may allow to understand how to extend the trajectories of the point-vortices after a collapse.

5.1.2 General properties of the α-point-vortex systems.

Hamiltonian formulation of the problem

The first and main property of the point-vortex systems is their Hamiltonian structure. Indeed, if we define the Hamiltonian of the system for $X = (x_1, \ldots, x_N)$ by

$$H(X) := \frac{1}{2}\sum_{i\neq j} a_i \, a_j \, G_\alpha\big(x_i - x_j\big), \tag{5.1.16}$$

then it is a direct computation from (5.1.14) to check that

$$a_i \frac{\mathrm{d}}{\mathrm{d}t} x_i(t) = \nabla^\perp_{x_i} H(X).$$

In particular, the Hamiltonian H is preserved by the flow associated to the differential equation (5.1.14). With this Hamiltonian structure of the equation, it is possible to make use of the Noether theorem to determine the other quantities that are left invariant. The invariance of the Hamiltonian with respect to the translations of the plane implies the preservation of the vorticity vector that is defined by

$$M(X) := \sum_{i=1}^{N} a_i \, x_i. \tag{5.1.17}$$

Similarly, the invariance of the Hamiltonian with respect to the rotations of the plane implies the preservation of the vorticity momentum defined by

$$I(X) := \sum_{i=1}^{N} a_i \, |x_i|^2 \tag{5.1.18}$$

These three conservation laws can also be obtained by a direct computation using the point-vortex equation (5.1.14) (see for instance [34]).

Non neutral clusters hypothesis

In the reference work by Marchioro and Pulvirenti [62, chap. 4], the authors study the generic situation for which the intensities of any cluster is not equal to 0 :

$$\forall\, P \subseteq \{1,\dots,N\} \; s.t.\; P \neq \emptyset, \qquad \sum_{i\in P} a_i \neq 0. \tag{5.1.19}$$

In this article, this non-degeneracy hypothesis will be referred as the "*non-neutral clusters hypothesis*" since it means that any vortex cluster is not neutral (in analogy with Coulomb interaction). The main reason behind such an hypothesis relies on the properties of the centers of vorticity. We denote by $\mathcal{P}(N)$ the collection of all the subsets of $\{1,\dots,N\}$ and we define

$$\mathcal{P}_0(N) := \mathcal{P}(N) \setminus \{\emptyset\}.$$

For $P \in \mathcal{P}_0(N)$, called in this article a *cluster of vortices*, we define the center of vorticity associated to the cluster P as being the barycenter :

$$B_P := \Big(\sum_{i\in P} a_i\Big)^{-1} \sum_{i\in P} a_i\, x_i. \tag{5.1.20}$$

If $P = \{1,\dots,N\}$, the barycenter is the center of vorticity of the whole system and it is simply denoted by B. As a consequence of the preservation of the center of vorticity (5.1.17), we have that B is preserved by the flow. For the center of vorticity of the clusters, similar computations gives the following quasi-conservation property :

$$\forall P \in \mathcal{P}_0(N), \qquad \left|\frac{\mathrm{d}}{\mathrm{d}t} B_P(t)\right| \le \sum_{i\in P}\sum_{j\notin P} \frac{C}{|x_i(t)-x_j(t)|^\alpha}, \tag{5.1.21}$$

where C is a constant depending on the intensities a_i. Indeed, using (5.1.14),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i\in P} a_i\, x_i(t) &= \sum_{i\in P}\sum_{j\notin P} a_i\, a_j\, \frac{(x_i(t)-x_j(t))^\perp}{|x_i(t)-x_j(t)|^{\alpha+1}} + \sum_{i\in P}\sum_{\substack{j\in P\\ j\neq i}} a_i\, a_j\, \frac{(x_i(t)-x_j(t))^\perp}{|x_i(t)-x_j(t)|^{\alpha+1}}\\ &= \sum_{i\in P}\sum_{j\notin P} a_i\, a_j\, \frac{(x_i(t)-x_j(t))^\perp}{|x_i(t)-x_j(t)|^{\alpha+1}},\end{aligned}$$

where for the last equality we used that $(x-y)^\perp = -(y-x)^\perp$ to cancel the second sum. The obtained equality eventually gives (5.1.21).

In [62, chap. 4] for Euler and in [34, Proposition 2.1] for the general case, the quasi-preservation of the centers of vorticity of the clusters is used as the main tool to prove the following uniform bound :

Theorem 5.1.1 (Uniform bound on the trajectories [34])**.** *Consider the point-vortex dynamic* (5.1.14) *under the non-neutral clusters hypothesis* (5.1.19) *and with a kernel profile* G_α *for* $\alpha \ge 0$ *fixed.*

Then, given any positive time $T > 0$*, there exists a constant* C *such that for any initial datum* $X \in \mathbb{R}^{2N}$ *that is not leading to a collapse on* $[0,T)$*,*

$$\sup_{t\in[0,T)} \left|X - S_\alpha^t X\right| \le C, \tag{5.1.22}$$

where S_α^t *is the flow associated to* (5.1.14) *with kernel profile* G_α*. Moreover, the constant* C *depends only on* N*, the intensities* a_i*, and on the final time* T*. This constant* C *does not depend on the initial datum* $X \in \mathbb{R}^{2N}$ *nor on* $\alpha \ge 0$*.*

The uniform relative bound theorem

A natural question concerning Theorem 5.1.1 is to ask what this result becomes when the non-neutral cluster hypothesis (5.1.19) ceases to be satisfied. In [34], the following hypothesis was considered :

$$\forall\, P \subseteq \{1,\dots,N\} \; s.t. \;\; P \neq \emptyset \;\text{ and }\; \{1,\dots,N\}, \qquad \sum_{i\in P} a_i \neq 0. \tag{5.1.23}$$

In other words, all the strict sub-clusters must have the sum of their intensities different from 0 but we allow the total sum $\sum_{i=1}^N a_i$ to be equal to 0. This situation is achieved for instance by the vortex pair of intensities $+1$ and -1 that is translating at a constant speed. Having a total vorticity equal to 0 corresponds to the following physical situation : the fluid is initially at rest and, at $t=0$, it is submitted to a vorticity-preserving perturbation.

Theorem 5.1.2 (Uniform relative bound on the trajectories [34, Proposition 2.3]). *For a given set of points noted $X=(x_1,\dots,x_N)\in\mathbb{R}^{2N}$, we define the diameter of this set by*

$$\text{diam}\,(X) \;:=\; \max_{i\neq j}|x_i-x_j|.$$

Consider the point-vortex dynamic (5.1.14) *under hypothesis* (5.1.23) *with a kernel profile G_α. Let $T>0$ the final time. Then, for all initial datum $X\in\mathbb{R}^{2N}$ that are not leading to collapse on $[0,T)$,*

$$\sup_{t\in[0,T)} \;\text{diam}\,\big(S_\alpha^t X\big) \leq \text{diam}\,(X) + C, \tag{5.1.24}$$

where S_α^t is the flow associated to (5.1.14)*. Moreover, the constant C depends only on N, the intensities a_i, and on the final time T. This constant C does not depend on the initial datum $X\in\mathbb{R}^{2N}$ nor on $\alpha>0$.*

The Hölder estimate that is proved in this article for the α-point-vortex system (enunciated hereafter) must be considered in comparison with these two theorems. More precisely, the theorem proved here can be seen in a certain way as an explicit computation of the constants appearing in (5.1.22) and in (5.1.24), and more precisely concerning the dependency of these constants with respect to the final time T. It is indeed a consequence of the theorem proved hereafter that the constant C in (5.1.22) can be replaced by $\overline{C}\,T^{\frac{1}{\alpha+1}}$ when $T\leq 1$, where $\overline{C}$ is a constant independent on the final time T, and on the initial position $X\in\mathbb{R}^{2N}$.

5.1.3 Main result of the article

Hölder regularity for point-vortices in the plane

The first result of the present article is about the α-point-vortex models in the whole plane under the non-neutral (sub)-clusters hypothesis :

Theorem 5.1.3 (Hölder regularity for α-point-vortex dynamics in the plane). *Consider the α-point-vortex dynamic* (5.1.14) *for a given $\alpha\geq 0$ with intensities $a_i\neq 0$. Consider an initial datum $X\in\mathbb{R}^{2N}$ such that the associated dynamic of point-vortices is well defined on $[0,T)$ for some time $T\in(0,1]$, with a possible collapse* (5.1.15) *at time T.*

(i) If the intensities satisfy the "non neutral clusters hypothesis" (5.1.19)*, then the trajectories of the point-vortices are Hölder continuous for all $i\in\{1,\dots,N\}$:*

$$\forall\, t_1<t_2\in[0,T), \qquad \big|x_i(t_2)\,-\,x_i(t_1)\big| \;\leq\; C\,|t_2-t_1|^{\frac{1}{\alpha+1}},$$

where the constant C depends only on N, α and on the intensities a_i. In particular, the trajectories converge as $t\to T^-$.

(ii) If the intensities satisfy only the "non neutral *sub*-clusters hypothesis" (5.1.23)*, then for all indices* $i \neq j \in \{1, \ldots, N\}$ *such that* $\liminf_{t\to T^-} |x_i(t) - x_j(t)| = 0$*, we have*

$$\forall\, t \in [0,T), \qquad \big|x_i(t) - x_j(t)\big| \;\leq\; C\,|T-t|^{\frac{1}{\alpha+1}},$$

where the constant C *depends only on* N*,* α*, and on the intensities* a_i*.*

(iii) If the intensities satisfy only the "non neutral *sub*-clusters hypothesis" (5.1.23) *and the dynamic is such that not all points collapse together, namely*

$$\max_{i\neq j}\ \limsup_{t\to T^-}\ |x_i(t) - x_j(t)| > 0, \tag{5.1.25}$$

then for all $i = 1 \ldots N$ *there exists* x_i^* *such that*

$$x_i(t) \longrightarrow x_i^*, \quad \text{as } t \to T^-.$$

Moreover, we have the following estimate at the time of collapse :

$$\forall\, t \in [0,T), \qquad \big|x_i(t) \,-\, x_i^*\big| \;\leq\; C'\,|T-t|^{\frac{1}{\alpha+1}},$$

for some constant C' *depending also on the initial data.*

Note that we imposed $T \leq 1$ so that we get a constant C that is independent on the final time for Cases (i) and (ii). This is not restrictive at all since the point-vortex system is autonomous with respect to time and then we can always reduce the study of the regularity of the system to this case.

The 1/2-Hölder regularity obtained in Case (i) of this theorem, is the same as obtained for the collapse of 3 vortices by [4] when $\alpha = 1$ (Euler case). When $\alpha = 2$ (SQG case with exponent $s = 1/2$) we also find the 1/3-Hölder regularity known for the 3-vortex problem [72]. In the general case, the Hölder exponent $1/(\alpha + 1)$ obtained here is the same as the one previously obtained in the partial result [33].

We prove in Appendix 5.4 the existence of self-similar collapses for all α and we observe that the regularity at the time of collapse is $\frac{1}{\alpha+1}$-Hölder and not better. This shows the optimality of the Hölder regularity obtained in the above theorem.

Hölder property for point-vortices in bounded domains

The second result of this article concerns the dynamics of Euler point-vortices in smooth bounded domains. We have a convergence result with Hölder regularity :

Theorem 5.1.4 (Hölder regularity for Euler point-vortices in bounded domains)**.** *Consider the Euler point-vortex dynamic* (5.1.8) *in a simply or multi-connected smooth domain* Ω*. Assume that the intensities* $a_i \neq 0$ *satisfy the* non neutral clusters hypothesis (5.1.19)*. Consider an initial datum* $X \in \mathbb{R}^{2N}$ *such that the dynamics is well-defined on* $[0,T)$ *with* $T > 0$*, with a possible collapse of point-vortices* (5.1.15) *at time* T*. Define*

$$I := \Big\{ i = 1 \ldots N \,:\, \liminf_{t\to T^-}\ \mathrm{dist}\,\big(x_i(t), \partial\Omega\big) = 0 \Big\}. \tag{5.1.26}$$

Then, there exists a constant C *such that the following properties hold true.*

(i) If $i \notin I$ *then*

$$\forall\, t_1 < t_2 \in [0,T), \qquad \big|x_i(t_2) - x_i(t_1)\big| \;\leq\; C\sqrt{t_2 - t_1}\,.$$

In particular, $x_i(t)$ *converges as* $t \to T^-$ *towards an interior point of* Ω*.*

(ii) If $i \in I$ *then*

$$\forall\, t_1 < t_2 \in [0,T), \qquad \Big|\mathrm{dist}\,\big(x_i(t_2), \partial\Omega\big) - \mathrm{dist}\,\big(x_i(t_1), \partial\Omega\big)\Big| \;\leq\; C\sqrt{t_2 - t_1}\,.$$

In particular, the distance between $x_i(t)$ *and* $\partial\Omega$ *converges to* 0 *as* $t \to T$*.*

Note that the Hölder regularity obtained for Euler in the plane (Theorem 5.1.3 with $\alpha = 1$) is the same as the one obtained for Euler in bounded domains (Theorem 5.1.4). Here the Hölder constant depends on the intensities a_i, the circulations ξ_m, Ω, N but also on the final time T and the initial data X, in a way that we will specify during the proof.

5.2 Proof of Theorem 5.1.3

This section is devoted to the proof of the Hölder estimate in the plane stated in Theorem 5.1.3. The proof relies heavily on the quasi conservation of the center of vorticity of the different clusters of vortices stated in (5.1.21). We first establish general properties for points x_i that evolves in $\mathbb{R}^p$ with $p \in \mathbb{N}^*$ under general assumptions. In a second step, we show that the point-vortex system satisfies these assumptions with $p = 2$ and this eventually leads to the conclusion.

5.2.1 Clusters of vortices

To start with, we define the degeneracy parameters

$$A_0 := \min_{\substack{P \in P(N) \\ P \neq \emptyset \\ P \neq \{1,\dots,N\}}} \left| \sum_{k \in P} a_k \right|,$$

and

$$A := \min_{\substack{P \in P(N) \\ P \neq \emptyset}} \left| \sum_{k \in P} a_k \right| = \min \left\{ A_0 ; \left| \sum_{i=1}^N a_i \right| \right\}. \tag{5.2.1}$$

Remark that $A_0 > 0$ is equivalent to the *non-neutral sub-clusters hypothesis* (5.1.23) and $A > 0$ is equivalent to the *non-neutral clusters hypothesis* (5.1.19). We also denote :

$$a_0 := \left| \sum_{i=1}^N a_i \right|, \qquad \text{and} \qquad a := \sum_{i=1}^N |a_i|. \tag{5.2.2}$$

We are in position to state the first lemma :

Lemma 5.2.1. *Let $(x_i)_{1 \le i \le N}$ be a family of $N \in \mathbb{N}^*$ points of $\mathbb{R}^p$ associated to intensities $(a_i)_{1 \le i \le N}$ satisfying the* non neutral sub cluster hypothesis (5.1.23). *Then for every $P \in P(N)$ such that $P \neq \{1, \dots, N\}$ and $P \neq \emptyset$, we have for all $i \in P$,*

$$|x_i - B_P| \le \frac{a}{A_0} \max_{j \in P} |x_i - x_j|,$$

where B_P was defined in (5.1.20).

Démonstration. The definition of B_P gives

$$x_i - B_P = \frac{\sum_{j \in P} a_j (x_i - x_j)}{\sum_{j \in P} a_j}.$$

Then, using the triangular inequality,

$$|x_i - B_P| \le \frac{\sum_{j \in P} |a_j| \, |x_i - x_j|}{\left| \sum_{j \in P} a_j \right|} \le \frac{\sum_{j \in P} |a_j|}{\left| \sum_{j \in P} a_j \right|} \max_{j \in P} |x_i - x_j|.$$

Recalling the definitions of a and A_0 given previously ends the proof. □

Remark 5.2.2. *In the previous Lemma, if we assume the stronger hypothesis* (5.1.19)*, then the conclusion is also true for* $P = \{1, \ldots, N\}$ *by replacing* A_0 *by* A*.*

To study the system of vortices, we separate them into clusters so that we can study separately the sets of vortices that are close to each-other. To build these clusters, we make use of the following lemma that is reminiscent of Lemma A.2 in [12].

Lemma 5.2.3 (Balls lemma)**.** *Let* $(x_i)_{1\le i\le N}$ *be* N *points in* $\mathbb{R}^p$*. Let* $\varepsilon > 0$ *and* $0 < \kappa \le \frac{1}{2}$*. We denote by* $\mathcal{B}(x, r)$ *the ball of* $\mathbb{R}^p$ *centered in* x *of radius* r*. Then there exist* $\delta > 0$ *and a set of indices* $Q \in \mathcal{P}_0(N)$ *such that*

$$\varepsilon \le \delta < \left(\frac{\kappa}{2}\right)^{-N} \varepsilon, \tag{5.2.3}$$

$$\bigcup_{i=1}^{N} \mathcal{B}(x_i, \varepsilon) \subset \bigcup_{j\in Q} \mathcal{B}(x_j, \delta) \tag{5.2.4}$$

and

$$\forall\, i \ne j \in Q, \quad |x_i - x_j| \ge \kappa^{-1}\delta. \tag{5.2.5}$$

Démonstration. First, define the set $Q_1 := \{1, \ldots, N\}$ and the parameter $\delta_1 := \varepsilon > 0$. Obviously, with these definitions, Property (5.2.4) is satisfied. We then proceed by iteration and build a sequence of sets Q_k (with $Q_k \subseteq Q_{k-1}$) for $k = 1 \ldots K$. We also define the parameters $\delta_k = \varepsilon(\kappa/2)^{-k+1}$. Suppose that the set Q_k has already been built for some k and that this set satisfies (5.2.4) with parameter δ_k, but does not satisfy (5.2.5). Then there exists $x_i \ne x_j$ in Q_k such that $|x_i - x_j| < \kappa^{-1}\delta_k$. We define $Q_{k+1} := Q_k \setminus \{j\}$ and we recall that $\delta_{k+1} = (\kappa/2)^{-1}\delta_k$. We next prove :

$$\bigcup_{\ell\in Q_k} \mathcal{B}(x_\ell, \delta_k) \subset \bigcup_{l\in Q_{k+1}} \mathcal{B}(x_\ell, \delta_{k+1})$$

which implies that Property (5.2.4) is still satisfied at step $k+1$. To prove the relation above, it suffices to establish that $\mathcal{B}(x_j, \delta_k) \subseteq \mathcal{B}(x_i, \delta_{k+1})$. This indeed holds true since for all $x \in \mathcal{B}(x_j, \delta_k)$ we have that $|x - x_i| \le |x - x_j| + |x_i - x_j| \le \delta_k + \kappa^{-1}\delta_k < (\kappa/2)^{-1}\delta_k = \delta_{k+1}$.

The sequence (Q_k) is such that Q_k has its cardinal equal to $N+1-k$. There exists necessarily a step $K \le N$ such that (5.2.5) holds. Indeed, if this is not the case for all $k = 1 \ldots N-1$, then at the N^{th} step the set Q_N is reduced to one element and thus (5.2.5) is satisfied since the condition is void. In particular, this construction requires at most N iterations and therefore Property (5.2.3) is satisfied with δ_k for all k. This eventually concludes the proof of Lemma 5.2.3 by setting $\delta := \delta_K$ and $Q := Q_K$. □

We deduce the following Corollary that we will use later in this article.

Corollary 5.2.4. *Let* (x_i) *be a family of* N *points in* $\mathbb{R}^p$*. Then for all* $\kappa \in (0, 1)$ *and for all* $d > 0$*, there exist* $\delta \in [\left(\frac{\kappa}{8}\right)^N d, d)$ *and a partition* $\mathfrak{P}$ *of the set* $\{1, \ldots, N\}$ *such that*

$$\forall\, P \in \mathfrak{P}, \quad \forall\, i, j \in P, \qquad |x_i - x_j| \le \delta \tag{5.2.6}$$

and

$$\forall\, P \ne P' \in \mathfrak{P}, \quad \forall\, i \in P, \quad \forall\, j \in P', \qquad |x_i - x_j| \ge \kappa^{-1}\delta. \tag{5.2.7}$$

Démonstration. Let $0 < \kappa < 1$. We set $\varepsilon = \frac{1}{2}\left(\frac{\kappa}{8}\right)^N d$ and $\kappa' = (2\kappa^{-1} + 2)^{-1}$. Then $\kappa' \in (0, \frac{1}{4})$ and we can apply Lemma 5.2.3 with ε and κ'. We then have a δ' and a set $Q \subseteq \{1, \ldots, N\}$ such that

$$\varepsilon \le \delta' < \left(\frac{\kappa'}{2}\right)^{-N} \varepsilon, \tag{5.2.8}$$

$$\bigcup_{i=1}^{N} \mathcal{B}(x_i, \varepsilon) \subset \bigcup_{j\in Q} \mathcal{B}(x_j, \delta') \tag{5.2.9}$$

and

$$\forall\, i \neq j \in Q, \quad |x_i - x_j| \geq (\kappa')^{-1}\delta'. \tag{5.2.10}$$

Let $\delta := 2\delta'$. Property (5.2.8) gives

$$\left(\frac{\kappa}{8}\right)^N d \leq \delta < \left(\frac{\kappa'}{2}\right)^{-N} \left(\frac{\kappa}{8}\right)^N d.$$

Since $\frac{\kappa}{\kappa'} = 2 + 2\kappa < 4$, we have $\delta \in [\left(\frac{\kappa}{8}\right)^N d, d)$.

For all $i \in Q$, we set

$$P_i = \left\{j \in \{1, \ldots, N\}, \quad |x_i - x_j| \leq \delta'\right\}.$$

Let $\mathfrak{P} = \{P_i,\, i \in Q\}$. The definition of P_i gives (5.2.6). Relation (5.2.10) and the definition of P_i imply that

$$\forall\, i_1 \neq i_2 \in Q,\, \forall\, j \in P_{i_1},\, \forall\, k \in P_{i_2}, \quad |x_j - x_k| \geq |x_{i_1} - x_{i_2}| - |x_j - x_{i_1}| - |x_k - x_{i_2}| \geq (\kappa')^{-1}\delta' - 2\delta' = \kappa^{-1}\delta,$$

which is (5.2.7). Finally, relation (5.2.9) implies that every index $i \in \{1, \ldots, N\}$ belongs to at least one element of $\mathfrak{P}$ and the relation above gives that the elements of $\mathfrak{P}$ are pairwise disjoints. Therefore $\mathfrak{P}$ is indeed a partition of $\{1, \ldots, N\}$. □

5.2.2 Sufficient condition to prevent a collapse

The Hölder regularity result can be seen as a necessary condition to have a collapse of point-vortices. The strategy is then to investigate the sufficient conditions that prevent a collapse. Within this approach, we proved the following general proposition :

Proposition 5.2.5 (Sufficient condition to prevent a collapse)**.** *For $1 \leq i \leq N$, let x_i be a family of N different points of $\mathbb{R}^p$ evolving on a time interval $[0, T)$, $T > 0$. We assume furthermore that*

$$\frac{\mathrm{d}}{\mathrm{d}t} x_i \in L^1_{\mathrm{loc}}([0, T)). \tag{5.2.11}$$

Let $(a_i)_{1\leq i\leq N} \in \mathbb{R}^N$ satisfy (5.1.23). Recall the definition of the center of vorticity of clusters with intensities a_i given at (5.1.20). We assume that these points evolve such that there exists $C_0, C_1 \geq 0$ and $\alpha \geq 0$ such that for all $t \in [0, T)$,

$$\forall P \in \mathcal{P}_0(N) \setminus \{1 \ldots N\}, \qquad \left|\frac{\mathrm{d}}{\mathrm{d}t} B_P(t)\right| \leq \sum_{i\in P} \sum_{j\notin P} \frac{C_0}{|x_i(t) - x_j(t)|^\alpha} + C_1. \tag{5.2.12}$$

Then there exists a constant $C_2 > 0$ such that for all $\eta \in (0, 1]$, for all $t \in [0, T)$ satisfying

$$T - t \leq C_2\, \eta^{\alpha+1},$$

and for all indices $i, j \in \{1, \ldots, N\}$, the following implication is true :

$$|x_i(t) - x_j(t)| \geq \eta \quad \Longrightarrow \quad \forall \tau \in [t, T), \quad |x_i(\tau) - x_j(\tau)| \geq \frac{\eta}{2}.$$

The constant C_2 depends only on α, a, A_0, C_0, C_1 and N.

The idea of the proof is the following. We group the points into clusters. We observe that past a certain time, if a point start moving in a significant manner, it means that its cluster has to spread and divide itself into smaller clusters. Indeed this would otherwise contradict the preservation of the center of vorticity. Since there is a finite number of points, spreading of clusters can happen only a finite number of times.

Démonstration. We can assume without loss of generality that $\max\{C_0, C_1\} > 0$. We fix once and for all some $\eta \in (0, 1]$. We also fix two indices $i \neq j$ and a time t_1. We assume that $|x_i(t_1) - x_j(t_1)| \geq \eta$ and

$$T - t_1 \leq C_2\, \eta^{\alpha+1} \tag{5.2.13}$$

for some constant C_2 to be chosen later. During the proof, we will impose several conditions on the constant C_2 and at the end of the proof we will observe that all these conditions can be satisfied for a constant C_2 which is independent of η.

Step 1 : Iterative construction of a sequence of clusters. We recall that A_0 and a are defined by (5.2.1) and (5.2.2) respectively. By hypothesis (5.1.23), we have that $A_0 > 0$. Let $0 < \kappa < \frac{A_0}{16a}$. Remark that $\kappa \leq 1/16$ since $A_0 \leq a$. We are now building partitions of $\{1, \dots, N\}$ with an iterative process.

We first invoke the corollary of the balls lemma (Corollary 5.2.4) to the points $x_k(t_1)$ to build the first partition $\mathfrak{P}^1$ by choosing $d := \kappa\eta$. This gives the first partition $\mathfrak{P}^1$ and a real number δ_1 satisfying

$$\left(\frac{\kappa}{8}\right)^N \kappa\eta \leq \delta_1 < \kappa\eta \tag{5.2.14}$$

such that

$$\forall\, P \in \mathfrak{P}^1, \quad \forall\, k, \ell \in P, \qquad |x_k(t_1) - x_\ell(t_1)| \leq \delta_1$$

and

$$\forall P \neq P' \in \mathfrak{P}^1, \quad \forall k \in P, \quad \forall \ell \in P', \quad |x_k(t_1) - x_\ell(t_1)| \geq \kappa^{-1}\delta_1.$$

The elements of such a partition are called *clusters*. Since $\delta_1 < \eta$, the indices i and j do not belong to the same cluster. In particular $\{1, \dots, N\} \notin \mathfrak{P}^1$. We now define

$$r := \min\left\{\frac{1}{8}\,;\, \frac{A_0}{8a\kappa} - 2\right\} > 0, \tag{5.2.15}$$

and

$$s := r\left(\frac{\kappa}{8}\right)^N.$$

Note that $0 < s < r < 1$.

We now build iteratively a finite number of partitions denoted by $\mathfrak{P}^q$, $q \geq 1$, a finite sequence of positive numbers δ_q and an increasing finite sequence of times (t_q) that satisfy the following properties :

$$\forall\, P \in \mathfrak{P}^q, \quad \forall\, k, \ell \in P, \qquad |x_k(t_q) - x_\ell(t_q)| \leq \delta_q. \tag{i}$$

and

$$\forall P \neq P' \in \mathfrak{P}^q, \quad \forall k \in P, \quad \forall \ell \in P', \quad |x_k(t_q) - x_\ell(t_q)| \geq \kappa^{-1}\delta_q. \tag{ii}$$

The construction proceeds as follows. If the following relation is satisfied :

$$\forall\, k = 1 \dots N, \quad \forall \tau \in [t_q, T), \qquad |x_k(\tau) - x_k(t_q)| \leq \kappa^{-1}\delta_q/8. \tag{$*$}$$

then the construction stops. Else, we will construct $t_{q+1} \in (t_q, T)$ such that

$$\forall\, k = 1 \dots N, \quad \forall \tau \in [t_q, t_{q+1}], \qquad |x_k(\tau) - x_k(t_q)| \leq \kappa^{-1}\delta_q/8. \tag{iii}$$

and δ_{q+1} such that

$$s\delta_q \leq \delta_{q+1} < r\delta_q. \tag{iv}$$

and the next partition $\mathfrak{P}^{q+1}$ satisfying (i) and (ii) at step $q+1$ and

$$\mathfrak{P}^{q+1} \text{ is a strict sub-partition of } \mathfrak{P}^q. \tag{v}$$

Notice that $\mathfrak{P}^1$ satisfies properties (i) and (ii). Let $q \in \mathbb{N}^*$ be fixed and assume now that the partitions $\mathfrak{P}^{q'}$ are built for all $q' = 1 \dots q$. Assuming that $(*)$ is not satisfied, we proceed to construct t_{q+1}, δ_{q+1} and $\mathfrak{P}^{q+1}$.

By continuity of the trajectories, we denote by $t_{q+1} \in [t_q, T)$ the largest time such that

$$\forall\, \tau \in [t_q, t_{q+1}], \quad \forall\, \ell = 1 \dots N, \qquad |x_\ell(\tau) - x_\ell(t_q)| \leq \kappa^{-1}\delta_q/8, \tag{5.2.16}$$

which is correctly defined since $(*)$ is not satisfied. Then (iii) holds true.

One uses now Corollary 5.2.4 with parameter $d = r\delta_q$ and with the family of points given by $x_i(t_{q+1})$ for $i = 1 \dots N$. This gives δ_{q+1} such that $\left(\frac{\kappa}{8}\right)^N d \leq \delta_{q+1} < d$ and a partition $\mathfrak{P}^{q+1}$ that satisfies

$$\forall\, P \in \mathfrak{P}^{q+1}, \quad \forall\, m, n \in P, \qquad |x_m(t_{q+1}) - x_n(t_{q+1})| \leq \delta_{q+1},$$

and

$$\forall\, P \neq P' \in \mathfrak{P}^{q+1}, \quad \forall m \in P, \quad \forall\, n \in P', \qquad |x_m(t_{q+1}) - x_n(t_{q+1})| \geq \kappa^{-1}\delta_{q+1}$$

which shows conditions (i) and (ii) at step $q+1$. Recalling that $d = r\delta_q$ and $s = \left(\frac{\kappa}{8}\right)^N r$ gives that $s\delta_q \leq \delta_{q+1} < r\delta_q$ so that condition (iv) is proved.

It remains to show condition (v). Using (ii) and (5.2.16), one has for any $P \in \mathfrak{P}^q$, for all $\tau \in [t_q, t_{q+1}]$, for any $m \in P$ and $n \notin P$ that

$$\begin{aligned}
|x_m(\tau) - x_n(\tau)| &= |x_m(\tau) - x_m(t_q) + x_m(t_q) - x_n(t_q) + x_n(t_q) - x_n(\tau)| \\
&\geq |x_m(t_q) - x_n(t_q)| - |x_m(\tau) - x_m(t_q)| - |x_n(t_q) - x_n(\tau)| \\
&\geq \kappa^{-1}\delta_q(1 - 2/8) \\
&\geq \kappa^{-1}\delta_q/2 \\
&> \delta_{q+1}
\end{aligned} \tag{5.2.17}$$

since $\kappa^{-1}/2 > 1 > r$ and $\delta_{q+1} < r\delta_q$ by (iv). This estimate applied at time $\tau = t_{q+1}$ implies that such two indexes m and n do not belong to the same cluster in $\mathfrak{P}^{q+1}$, otherwise it contradicts (i). Therefore two indexes belonging to two different clusters in $\mathfrak{P}^q$ still belong to two different clusters in $\mathfrak{P}^{q+1}$. This proves that $\mathfrak{P}^{q+1}$ is a sub-partition of $\mathfrak{P}^q$.

We now prove that it is a strict sub-partition. Since t_{q+1} is the largest time such that (5.2.16) holds, there exists at least one index $k \in \{1, \dots, N\}$ such that

$$|x_k(t_{q+1}) - x_k(t_q)| = \kappa^{-1}\delta_q/8. \tag{5.2.18}$$

Let $P \in \mathfrak{P}^q$ such that $k \in P$. Note that since $\mathfrak{P}^1$ is not the trivial partition $\{\{1, \dots, N\}\}$, neither is $\mathfrak{P}^q$ since $\mathfrak{P}^q$ is a sub-partition of $\mathfrak{P}^1$ as shown above. Thus, $P \neq \{1, \dots, N\}$. Therefore thanks to Hypothesis (5.1.23) we can apply Lemma 5.2.1 to obtain :

$$|x_k(t_{q+1}) - B_P(t_{q+1})| \leq \frac{a}{A_0} \max_{\ell \in P} |x_k(t_{q+1}) - x_\ell(t_{q+1})|, \tag{5.2.19}$$

and

$$|x_k(t_q) - B_P(t_q)| \leq \frac{a}{A_0} \max_{\ell \in P} |x_k(t_q) - x_\ell(t_q)| \leq \frac{a}{A_0}\delta_q, \tag{5.2.20}$$

where for the last inequality one uses the recursive hypothesis (i). Recall here hypothesis (5.2.12) :

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} B_P(t)\right| \leq \sum_{m \in P} \sum_{n \notin P} \frac{C_0}{|x_m(t) - x_n(t)|^\alpha} + C_1. \tag{5.2.21}$$

Recall that by relation (5.2.17), if $P \neq P' \in \mathfrak{P}^q$, then for $m \in P$, $n \in P'$ and $\tau \in [t_q, t_{q+1}]$ we have

$$|x_m(\tau) - x_n(\tau)| \geq \kappa^{-1}\delta_q/2.$$

Plugging this into (5.2.21) gives for all $\tau \in [t_q, t_{q+1}]$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}B_P(\tau)\right| \leq \frac{2^\alpha\, N^2\, C_0}{(\kappa^{-1}\delta_q)^\alpha} + C_1. \tag{5.2.22}$$

Thus,

$$|B_P(t_{q+1}) - B_P(t_q)| \leq \left(\frac{2^\alpha\, N^2\, C_0}{(\kappa^{-1}\delta_q)^\alpha} + C_1\right)(t_{q+1} - t_q). \tag{5.2.23}$$

We assume that the constant C_2 is small enough such that

$$C_2\eta^{\alpha+1} \leq \frac{a}{A_0}\delta_q\left(\frac{2^\alpha\, N^2\, C_0}{(\kappa^{-1}\delta_q)^\alpha} + C_1\right)^{-1}. \tag{5.2.24}$$

Combining this with relation (5.2.13), we have that

$$|T - t_1| \leq C_2\eta^{\alpha+1} \leq \frac{a}{A_0}\delta_q\left(\frac{2^\alpha\, N^2\, C_0}{(\kappa^{-1}\delta_q)^\alpha} + C_1\right)^{-1}.$$

Thus, since $t_1 \leq t_q \leq t_{q+1} < T$,

$$|t_q - t_{q+1}| \leq \frac{a}{A_0}\delta_q\left(\frac{2^\alpha\, N^2\, C_0}{(\kappa^{-1}\delta_q)^\alpha} + C_1\right)^{-1}.$$

Therefore, plugging this in (5.2.23),

$$|B_P(t_{q+1}) - B_P(t_q)| \leq \frac{a}{A_0}\delta_q. \tag{5.2.25}$$

Gathering now (5.2.18), (5.2.20) and (5.2.25) gives

$$\begin{aligned}|x_k(t_{q+1}) - B_P(t_{q+1})| &= |x_k(t_{q+1}) - x_k(t_q) + x_k(t_q) - B_P(t_q) + B_P(t_q) - B_P(t_{q+1})|\\ &\geq |x_k(t_{q+1}) - x_k(t_q)| - |x_k(t_q) - B_P(t_q)| - |B_P(t_q) - B_P(t_{q+1})|\\ &\geq (\kappa^{-1}/8 - 2a/A_0)\delta_q.\end{aligned}$$

The estimate (5.2.19) yields the fact that there exists an index $\ell \in P$ such that

$$|x_k(t_{q+1}) - x_\ell(t_{q+1})| \geq \frac{A_0}{a}|x_k(t_{q+1}) - B_P(t_{q+1})|$$

and combining this with the previous estimates gives

$$|x_k(t_{q+1}) - x_\ell(t_{q+1})| \geq \frac{A_0}{a}(\kappa^{-1}/8 - 2a/A_0)\delta_q \geq r\,\delta_q, \tag{5.2.26}$$

since r is defined by (5.2.15). Since $r\delta_q > 0$, this implies in particular that $k \neq \ell$ and therefore P has at least two elements.

Using $\delta_{q+1} < r\delta_q$ and relation (5.2.26), we have that $\delta_{q+1} < |x_k(t_{q+1}) - x_\ell(t_{q+1})|$. Thus the two indices k and ℓ do not belong to the same cluster anymore in $\mathfrak{P}^{q+1}$, since (i) is satisfied at step $q+1$. Consequently, $\mathfrak{P}^{q+1}$ is a strict sub-partition of $\mathfrak{P}^q$. All the recursive properties are proved. This concludes our construction.

We observe now that at some step Q condition $(*)$ must be verified. Indeed, since at every step the new partition is a strict sub-partition of the previous one, the number of clusters in the partition is a strictly increasing sequence, starting from at least 2 elements and cannot exceed

N. We insist on the fact that we proved that if $(*)$ is not satisfied then we *can* construct the next strict sub-partition. Since it is not possible to iterate more than $N-2$ times, relation $(*)$ is achieved after at most $N-2$ steps. We thus have $Q \le N-1$, with condition $(*)$ satisfied at step Q.

Step 2 : conclusion of the proof of the Proposition. We now prove that for every $\tau \in [t_1, T)$, $|x_i(\tau) - x_j(\tau)| \ge \eta/2$.

Let us start by noticing that hypothesis (iv) of our construction gives by induction that for all $q \in \{1, \ldots, Q\}$ we have

$$s^{q-1}\delta_1 \le \delta_q \le r^{q-1}\delta_1.$$

Recalling that δ_1 satisfies (5.2.14), and the fact that $s < \left(\frac{\kappa}{8}\right)^N$, we have that

$$s^q \kappa\eta \le \delta_q \le r^{q-1}\kappa\eta. \tag{5.2.27}$$

Let $\tau \in [t_1, T)$, and we set $t_{Q+1} = T$ so that there exists a unique $q \in \{1, Q\}$ verifying $\tau \in [t_q, t_{q+1})$. We have that for all $k \in \{1, \ldots, N\}$,

$$|x_k(\tau) - x_k(t_1)| \le \sum_{q'=1}^{q-1} |x_k(t_{q'+1}) - x_k(t_{q'})| + |x_k(\tau) - x_k(t_q)|.$$

Then, by the construction hypothesis (iii) and $(*)$ we have that

$$|x_k(\tau) - x_k(t_1)| \le \sum_{q'=1}^{q} \kappa^{-1}\delta_{q'}/8.$$

Recall from (5.2.15) that $r < 1/8$. By relation (5.2.27) this yields for all k and for all $\tau \ge t_1$,

$$|x_k(\tau) - x_k(t_1)| \le \frac{1}{8}\sum_{q'=1}^{q} r^{q'-1}\eta \le \eta \sum_{q'=1}^{q} \left(\frac{1}{8}\right)^{q'} \le \eta/4.$$

Therefore,

$$\begin{aligned}
|x_i(\tau) - x_j(\tau)| &= |x_i(\tau) - x_i(t_1) + x_i(t_1) - x_j(t_1) + x_j(t_1) - x_j(\tau)| \\
&\ge |x_i(t_1) - x_j(t_1)| - |x_i(\tau) - x_i(t_1)| - |x_j(t_1) - x_j(\tau)| \\
&\ge \eta - 2\eta/4 \\
&\ge \eta/2.
\end{aligned}$$

To conclude the proof of Proposition 5.2.5, one still has to check that C_2 can be chosen independently of η. The constant C_2 must satisfy Condition (5.2.24) for all indices $q \in \{1, \ldots, Q\}$. This leads to the following condition :

$$C_2\eta^{\alpha+1} \le \min_{q\in\{1,\ldots,Q\}} \frac{a}{A_0}\delta_q \left(\frac{2^\alpha N^2 C_0}{(\kappa^{-1}\delta_q)^\alpha} + C_1\right)^{-1}. \tag{5.2.28}$$

Since δ_q is of size η and it is the only quantity depending on η, we observe that both sides of this relation are of size $\eta^{\alpha+1}$. So the existence of C_2 independent of η is clear. More precisely, since $\kappa^{-1}\delta_q < \eta \le 1$ by (5.2.27) and by the fact that $r \le 1$, we have that

$$C_1 \le \frac{C_1}{(\kappa^{-1}\delta_q)^\alpha}.$$

Therefore, it is enough to take C_2 such that

$$C_2\eta^{\alpha+1} \le \left(\min_{q\in\{1,\ldots,Q\}} \delta_q^{\alpha+1}\right) \frac{a}{A_0}\frac{\kappa^{-\alpha}}{2^\alpha N^2 C_0 + C_1}.$$

Using relation (5.2.27) and the fact that $Q \leq N-1$, we have that

$$\delta_q^{\alpha+1} \geq (s^{N-1}\kappa\eta)^{\alpha+1}.$$

This proves that if we chose

$$C_2 = \frac{a}{A_0} s^{(N-1)(\alpha+1)} \frac{\kappa}{2^\alpha N^2 C_0 + C_1}$$

then relations (5.2.24) are satisfied for all $q = 1 \dots Q$. Replacing s by its value and taking for example $\kappa = \frac{A_0}{17a}$ eventually gives the announced constant C_2 in Theorem 5.1.3. Note that C_2 is positive and is finite since $\max\{C_0, C_1\} > 0$. This constant is given explicitly and depends only on a, A_0, α, C_0, C_1 and N. □

5.2.3 Conclusion of the proof

Convergence lemmas

With this proposition at hand, we are in position to state the lemmas which will allow to conclude the proof of Theorem 5.1.3. We first prove the following convergence result on the relative dynamics $t \mapsto x_i(t) - x_j(t)$:

Lemma 5.2.6 (Clusters of points going to collision)**.** *Let $x_i(t) \in \mathbb{R}^p$ be a set of points evolving in time on a time interval $[0,T)$. We assume that for all $t \in [0,T)$ and $i \neq j$, $x_i(t) \neq x_j(t)$ and that relation* (5.2.11) *is satisfied. Consider intensities a_i associated to these points such that the non-neutral sub-clusters hypothesis* (5.1.23) *holds true. Assume that we have* (5.2.12) *for some $\alpha \geq 0$. Then there exists a partition $\mathfrak{P}$ of the set $\{1,\dots,N\}$ and a distance $\delta > 0$ such that*

$$\forall P \in \mathfrak{P}, \quad \forall\, i,j \in P, \qquad \lim_{t\to T^-} |x_i(t) - x_j(t)| = 0 \tag{5.2.29}$$

and

$$\forall P \neq P' \in \mathfrak{P}, \quad \forall i \in P, \quad \forall j \in P', \quad \forall t \in [0,T), \qquad |x_i(t) - x_j(t)| \geq \delta. \tag{5.2.30}$$

Démonstration. We observe that the relation $i \sim j$ if $\lim_{t\to T^-} |x_i(t) - x_j(t)| = 0$ is an equivalence relation. We define the partition $\mathfrak{P}$ as the equivalence classes associated to this equivalence relation. Then relation (5.2.29) is trivially verified. We prove now that relation (5.2.30) is also true.

It suffices to show that if there are two indices $i \neq j$ such that

$$\liminf_{t\to T^-} |x_i(t) - x_j(t)| = 0,$$

then

$$\lim_{t\to T^-} |x_i(t) - x_j(t)| = 0.$$

Assume that $\liminf_{t\to T^-} |x_i - x_j| = 0$ and for the sake of contradiction that $|x_i - x_j|$ doesn't converge to 0. We infer that there exists $\eta \in (0,1]$ and two sequences $t_n < t'_n < T$ of times going to T as n goes to infinity such that

$$\begin{cases} |x_i(t_n) - x_j(t_n)| \geq \eta \\ |x_i(t'_n) - x_j(t'_n)| < \eta/2. \end{cases}$$

According to Proposition 5.2.5, this cannot hold as soon as $T - t_n \leq C_2 \eta^{\alpha+1}$. We have our contradiction. □

The elements of $\mathfrak{P}$ are called the *clusters of collisions* since i, j belong to the same P if and only if the associated point-vortices $x_i(t)$ and $x_j(t)$ are going to collide. With this convergence property, we can state the final lemma :

Lemma 5.2.7 (Hölder estimate lemma). *Let $x_i(t) \in \mathbb{R}^p$ be a set of points evolving on an interval of time $[0, T)$ with $T \leq 1$, satisfying* (5.2.11) *and that for all $t \in [0, T)$ and $i \neq j$, $x_i(t) \neq x_j(t)$. Associate to these points their intensities a_i such that the non-neutral sub-clusters hypothesis* (5.1.23) *holds true. Let $\alpha \geq 0$ and assume that the system satisfies* (5.2.12) *with some constants C_0, C_1. Let $\mathfrak{P}$ be the cluster partition defined by Lemma 5.2.6. We have the following properties.*

(i) For all $P \in \mathfrak{P}$ and for all $i, j \in P$ we have that

$$\forall\, t \in [0, T), \qquad \big|x_i(t) - x_j(t)\big| \;\leq\; C|T - t|^{\frac{1}{\alpha+1}}, \tag{5.2.31}$$

with a constant C that depends only on the a_i, N, C_0, C_1 and on α.

(ii) Assume in addition that $\sum_{i=1}^N a_i \neq 0$ and that relation (5.2.12) *holds true also for $P = \{1, \ldots, N\}$. Then for all $i = 1 \ldots N$ there exists $x_i^* \in \mathbb{R}^2$ such that*

$$x_i(t) \;\longrightarrow\; x_i^*, \qquad \text{as } t \to T. \tag{5.2.32}$$

Moreover,

$$\forall\, t \in [0, T), \qquad \big|x_i(t) - x_i^*\big| \;\leq\; C|T - t|^{\frac{1}{\alpha+1}}, \tag{5.2.33}$$

with a constant C that depends only on the a_i, N, C_0, $C_1$and on α.

(iii) Consider now the case $\sum_{i=1}^N a_i = 0$ and $\mathfrak{P} \neq \{\{1, \ldots, N\}\}$. Relations (5.2.32) *and* (5.2.33) *still hold true but here the constant C depends also on the δ given by* (5.2.30).

Démonstration. We start with Case (i). Let $P \in \mathfrak{P}$ be the cluster partition defined by Lemma 5.2.6 and $i, j \in P$. Let $t < T$ and let us define $\eta := |x_i(t) - x_j(t)|$. We consider two cases : $\eta \leq 1$ and $\eta > 1$.

Assume first that $\eta \leq 1$. The collision condition (5.2.29), in view of Proposition 5.2.5, gives that $T - t > C_2\, \eta^{\alpha+1}$. Indeed, otherwise Proposition 5.2.5 would imply that $|x_i(\tau) - x_j(\tau)| \geq \eta/2$ for all $\tau \geq t$. This gives that x_i and x_j do not collide, which contradicts the definition of $\mathfrak{P}$ from Lemma 5.2.6 (recall that $i, j \in P \in \mathfrak{P}$). In other words, when i, j belong to the same cluster of collapse P, we have that

$$T - t > C_2\, |x_i(t) - x_j(t)|^{\alpha+1}.$$

This is enough to get (5.2.31). It is a direct consequence of Proposition 5.2.5 that the constant appearing above depends only on N, a_i C_0, C_1 and α.

We consider now the case $\eta > 1$. Since we are in a cluster of collapse, there is a time $t' > t$ such that the distance between the vortices is equal to 1, by continuity of the trajectories and (5.2.29). We then apply Proposition 5.2.5 similarly as above but at time t' to get

$$T - t \geq T - t' \geq C_2\, |x_i(t') - x_j(t')|^{\alpha+1} = C_2. \tag{5.2.34}$$

We also know that the relative positions are uniformly bounded by Theorem 5.1.2. We denote by C_3 the constant given by this theorem for intervals of time smaller than 1 (the constant given by Theorem 5.1.2 may blow-up as $T \to +\infty$, this is the reason why we assume $T \leq 1$). More precisely, we make use of a slightly better result than Theorem 5.1.2, that is given by Proposition 4.1 in [34]. This proposition states in particular that [1] there exists a constant C_3 depending only on N, the intensities a_i, C_0, and C_1 such that

$$\forall\, t_1 < t_2 \in [0, T), \qquad \Big|\big(x_i(t_2) - x_j(t_2)\big) - \big(x_i(t_1) - x_j(t_1)\big)\Big| \leq C_3. \tag{5.2.35}$$

1. Proposition 4.1 in [34] is only stated for the point-vortex system but the only property of the point-vortices that is used is (5.2.12) so that the bound (5.2.35) remain true in the general case studied here.

In the equation above we take $t_1 = t$ and we let $t_2 \to T$. Using (5.2.29), this gives $|x_i(t) - x_j(t)| \leq C_3$. Therefore, with (5.2.34),

$$|x_i(t) - x_j(t)|^{\alpha+1} \leq C_3^{\alpha+1} \leq \frac{C_3^{\alpha+1}}{C_2}(T-t).$$

This concludes the proof of (5.2.31) and then of Case (i).

We now go to Case (ii). We start by proving that the trajectories converge as $t \to T$. Since $\sum_{i=1}^N a_i \neq 0$, we have that the non neutral clusters hypothesis (5.1.19) holds true and thus we can apply Lemma 5.2.1 (see also Remark 5.2.2) to get that for any $P \in \mathfrak{P}$ and for every $i \in P$,

$$|x_i(t) - B_P(t)| \leq \frac{a}{A} \max_{j \in P} |x_i(t) - x_j(t)|.$$

This and relation (5.2.31) imply that

$$|x_i(t) - B_P(t)| \leq \frac{a}{A} C\,|T-t|^{\frac{1}{\alpha+1}}. \tag{5.2.36}$$

Finally, by hypothesis, B_P satisfies (5.2.12). By Lemma 5.2.6 we know that relation (5.2.30) holds so that (5.2.12) becomes

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} B_P(t)\right| \leq N^2 \frac{C_0}{\delta^\alpha} + C_1. \tag{5.2.37}$$

This means that $t \mapsto B_P(t)$ itself is uniformly Lipschitz (with a constant depending on δ) and therefore converges as $t \to T$. This fact together with (5.2.36) proves that x_i is converging towards some x_i^*. This also gives a rate of convergence however the Hölder constant depends on δ because of relation (5.2.37). In order to prove that this constant does not depend on δ, we use an additional argument.

We set $X^* := \{x_i^* : i = 1 \dots N\}$. Up to an omitted modification of the labels of the indices for the points x_i, there exists $K \leq N$ such that

$$X^* = \{x_i^* : i = 1 \dots K\}, \qquad \text{and} \qquad \forall\, i \neq j \in \{1, \dots, K\}, \quad x_i^* \neq x_j^*.$$

We introduce the evolution system ζ_k with $k = 1 \dots N+K$ defined by

$$\zeta_i(t) := x_i(t), \quad \text{for } i = 1 \dots N \qquad \text{and} \qquad \zeta_{N+i}(t) := x_i^*, \quad \text{for } i = 1 \dots K.$$

The intensities b_k associated to this system are defined in the following manner. If $k \leq N$ we set $b_k = a_k$. If $k \geq N+1$, the b_k are chosen such that the non-neutral clusters hypothesis (5.1.19) holds true for $(a_i)_{1 \leq i \leq N+K}$. Note that such a choice is always possible since we are in the case where the a_i satisfy (5.1.19). We now compute for $P \subseteq \{1, \dots, N+K\}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} B_P = \Big(\sum_{k \in P} b_k\Big)^{-1} \sum_{k \in P} b_k \frac{\mathrm{d}\zeta_k}{\mathrm{d}t} = \Big(\sum_{k \in P} b_k\Big)^{-1} \sum_{\substack{i \in P \\ i \leq N}} a_i \frac{\mathrm{d}x_i}{\mathrm{d}t}.$$

If $P \cap \{1, \dots, N\} \neq \emptyset$ then $\frac{\mathrm{d}}{\mathrm{d}t} B_P = 0$. Else $P \cap \{1, \dots, N\} \neq \emptyset$ and we infer that since the dynamics of the $(x_i)_{1 \leq i \leq N}$ satisfies (5.2.12),

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}t} B_P\right| &= \left|\Big(\sum_{k \in P} b_k\Big)^{-1} \Big(\sum_{\substack{i \in P \\ i \leq N}} a_i\Big) \frac{\mathrm{d}}{\mathrm{d}t} B_{P \cap \{1, \dots, N\}}\right| \\
&\leq \sum_{\substack{i \in P \\ i \leq N}} \sum_{\substack{j \notin P \\ j \leq N}} \frac{C_0'}{|x_i(t) - x_j(t)|^\alpha} + C_1' \\
&\leq \sum_{k \in P} \sum_{\ell \notin P} \frac{C_0'}{|\zeta_k(t) - \zeta_\ell(t)|^\alpha} + C_1',
\end{aligned}$$

where

$$C_0' := C_0 \left| \Big(\sum_{k \in P} b_k \Big)^{-1} \sum_{\substack{i \in P \\ i \leq N}} a_i \right| \qquad \text{and} \qquad C_1' := C_1 \left| \Big(\sum_{k \in P} b_k \Big)^{-1} \sum_{\substack{i \in P \\ i \leq N}} a_i \right|.$$

Therefore, the system of points $\zeta_k(t)$ satisfies hypothesis (5.2.12), and then the conclusions of Lemma 5.2.6 and of Lemma 5.2.7-(i) hold true. In particular, if we take $i \in \{1, \dots, N\}$ and if we take $k \in \{1, \dots, K\}$ such that $x_i^* = x_k^*$, we have that ζ_i and ζ_{N+k} belong to the same cluster of collapse. We can then apply Lemma 5.2.7-(i) to the dynamics of the ζ_k and relation (5.2.31) for ζ_i and ζ_{N+k} to get that

$$\big| x_i(t) - x_i^* \big| \leq C |T - t|^{\frac{1}{\alpha+1}},$$

with a constant C that depends only on N, α, C_0, C_1 and on the a_i.

We end the proof with Case (iii). We assume that $\mathfrak{P} \neq \{\{1, \dots, N\}\}$. Then any $P \in \mathfrak{P}$ is different from $\{1, \dots, N\}$. Then the same argument as in Case (ii) shows that (5.2.36) and (5.2.37) still hold true with A replaced by A_0. These two relations give (5.2.33), with a constant C that this time depends on δ too. This completes the proof of the lemma. □

We underline that Case (i) in Lemma 5.2.7 above gives a constant C that does not depend on δ, unlike Case (iii). Nevertheless, the first case only gives properties on the relative dynamics. In a sense, the first case can be seen as the limit case when $\delta \to 0$.

End of the proof of Theorem 5.1.3

To end the proof of Theorem 5.1.3, we note that thanks to relation (5.1.21) the point-vortex dynamics (5.1.14) satisfies (5.2.11) and (5.2.12) with $C_1 = 0$. We can then apply Lemma 5.2.7 with $p = 2$ (since the point-vortices evolve in the plane) to get the announced Hölder regularity. More precisely, if $\mathfrak{P}$ is the set of clusters of collisions given by Lemma 5.2.6, we have :

Proof of Theorem 5.1.3-(i) : This case is a direct consequence of Lemma 5.2.7-(ii) applied on the time interval $[0, t_2]$, that is by making the time t_2 play the role of the final time. This is possible since the Hölder constant does not depend neither on $T \leq 1$ nor on δ.

Proof of Theorem 5.1.3-(ii) : This case is only a reformulation Lemma 5.2.7-(i).

Proof of Theorem 5.1.3-(iii) : If Condition (5.1.25) holds, then Lemma 5.2.7-(iii) applies. Indeed, one can check directly that the case where $\mathfrak{P} = \{\{1, \dots, N\}\}$ is exactly the negation of Condition (5.1.25). We conclude that Condition (5.1.25) implies that the trajectories of the vortices are convergent at the time of collapse $t = T$. Nevertheless, in this case the constant depends on the distance $\delta > 0$ and this distance δ depends directly on the initial datum $X \in \mathbb{R}^{2N}$ (the equation being deterministic).

The proof of Theorem 5.1.3 is completed. □

We are not able in Case (iii) to establish the Hölder regularity on the time interval $[0, T)$. This would require to apply Lemma 5.2.7-(iii) on the time interval $[0, t_2]$, as in Case (i). If we do that, since there is no collapse at time t_2, the cluster partitions is formed of singletons. Then the constant δ at time t_2 is

$$\delta(t_2) = \min_{i \neq j} \min_{t \in [0, t_2]} |x_i(t) - x_j(t)|.$$

Since the Hölder constant depends on $1/\delta(t_2)$ and $\delta(t_2) \to 0$ as $t_2 \to T$ (if there is a collapse at time T), this Hölder constant can not be chosen uniformly in t_2.

5.3 Proof of Theorem 5.1.4

In this section we are interested in the dynamics of vortices that lay in Ω, a smooth bounded domain. We assume for all this section that the intensities satisfy the non neutral cluster hypothesis (5.1.19).

The proof of the convergence property for the vortices in Ω is inspired from an analogous previous result for the point-vortex model in the plane [34]. The Hölder regularity makes use of Proposition 5.2.5. For the vortices going to the boundary in finite time, we will first prove convergence of the distance to the boundary, then establish the Hölder regularity with the use of Proposition 5.3.6, similar to Proposition 5.2.5.

Let $(t \mapsto x_i(t))_i$ be a solution of the Euler point-vortex problem (5.1.8) in Ω and assume that there is no collapse on the time interval $[0, T)$. For all $t \in [0, T)$ define the distribution $P_t \in \mathcal{D}'(\Omega)$ by

$$P_t := \sum_{i=1}^{N} a_i \, \delta_{x_i(t)}. \tag{5.3.1}$$

The proof of Theorem 5.1.4 is separated into three parts. We first study the distribution P_t for all $t \in [0, T)$ and prove that it converges as $t \to T$. Secondly, we use this result to obtain the convergence of the position of the vortices that have an accumulation point inside Ω. In the last part, we use this convergence result to prove the announced Hölder regularity results with arguments similar to the full plane case (Theorem 5.1.3).

5.3.1 Evolution in time of the studied distribution

Lemma 5.3.1. *Let φ be $\mathcal{C}^\infty$ and compactly supported inside Ω and let P_t be the distribution on Ω defined by* (5.3.1)*. Then,*

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \langle P_t, \varphi \rangle \right| \; \leq \; C \|\nabla \varphi\|_{L^\infty} \left(\frac{1}{\operatorname{dist}(\operatorname{supp}(\varphi), \partial\Omega)} + 1 \right) + C \, \|\nabla^2 \varphi\|_{L^\infty},$$

where supp(φ) is the support of φ and where

$$\operatorname{dist}(A, B) := \inf_{x \in A} \, \inf_{y \in B} \, |x - y|.$$

The constant C depends on the coefficients a_i, on the circulations ξ_m and on Ω.

Démonstration. A direct computation using the evolution equations for the point-vortex problem in bounded domains under developed form (5.1.11) gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \langle P_t, \varphi \rangle &= \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{N} a_i \varphi\big(x_i(t)\big) = \sum_{i=1}^{N} a_i \nabla \varphi\big(x_i(t)\big) \cdot \frac{\mathrm{d}}{\mathrm{d}t} x_i(t) \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} a_i \, a_j \nabla \varphi\big(x_i(t)\big) \cdot \nabla_x^\perp \gamma_\Omega\big(x_i(t), x_j(t)\big) \\
&\quad + \frac{1}{2\pi} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} a_i \, a_j \, \nabla \varphi\big(x_i(t)\big) \cdot \nabla^\perp G_1\big(x_i(t) - x_j(t)\big) \\
&\quad + \sum_{i=1}^{N} a_i \nabla \varphi\big(x_i(t)\big) \cdot \Big(\sum_{m=1}^{M} c_m(t) \beta_m\big(x_i(t)\big) \Big) \\
&:= S_1 + S_2 + S_3.
\end{aligned} \tag{5.3.2}$$

The first sum appearing on the right-hand side of (5.3.2), noted S_1, is estimated term by term using (5.1.6) :

$$\forall\, i = 1 \dots N, \qquad \Big|\nabla\varphi\big(x_i(t)\big) \cdot \nabla_x^\perp \gamma_\Omega\big(x_i(t), x_j(t)\big)\Big| \;\le\; C\frac{\big|\nabla\varphi\big(x_i(t)\big)\big|}{\operatorname{dist}\big(x_i(t), \partial\Omega\big)}. \tag{5.3.3}$$

One continues the estimate of (5.3.3) using that the function φ is compactly supported inside Ω :

$$\Big|\nabla\varphi\big(x_i(t)\big) \cdot \nabla_x^\perp \gamma_\Omega\big(x_i(t), x_i(t)\big)\Big| \;\le\; C \begin{cases} \dfrac{\|\nabla\varphi\|_{L^\infty}}{\operatorname{dist}\big(x_i(t), \partial\Omega\big)} & \text{if } x_i(t) \in \operatorname{supp}(\varphi), \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$|S_1| \;\le\; C\,\frac{\|\nabla\varphi\|_{L^\infty}}{\operatorname{dist}(\operatorname{supp}(\varphi), \partial\Omega)}.$$

To estimate the second term in (5.3.2), denoted by S_2, one proceeds to a symmetrization of the double sum by swapping the indices $i \leftrightarrow j$. This gives

$$S_2 = \frac{1}{4\pi}\sum_{i=1}^N \sum_{\substack{j=1\\ j\neq i}}^N a_i\, a_j \Big(\nabla\varphi\big(x_i(t)\big) - \nabla\varphi\big(x_j(t)\big)\Big) \cdot \nabla^\perp G_1\big(x_i(t) - x_j(t)\big).$$

Using relation (5.1.3),

$$\begin{aligned} &\Big|\Big(\nabla\varphi\big(x_i(t)\big) - \nabla\varphi\big(x_j(t)\big)\Big) \cdot \nabla^\perp G_1\big(x_i(t) - x_j(t)\big)\Big| \\ &\qquad \le \frac{\big|\nabla\varphi\big(x_i(t)\big) - \nabla\varphi\big(x_j(t)\big)\big|}{|x_i(t) - x_j(t)|} \le \|\nabla^2\varphi\|_{L^\infty}, \end{aligned}$$

where the last inequality is simply the mean value theorem applied to the function $\nabla\varphi$. Thus, $|S_2| \le C\|\nabla^2\varphi\|_{L^\infty}$. Concerning the last term S_3, since the functions β_m and c_m are bounded by standard elliptic estimates, we have that $|S_3| \le C\|\nabla\varphi\|_{L^\infty}$. Gathering these three estimates back into (5.3.2) concludes the proof. □

Now that we obtained an upper bound on the derivative of this distribution we study its limit as $t \to T$.

Lemma 5.3.2. *Let $(t \mapsto x_i(t))_i$ be a solution of the Euler point-vortex problem in bounded domains* (5.1.8)*. Define the distribution P_t with* (5.3.1)*. Then for all $i = 1 \dots N$, there exist $x_i^* \in \overline{\Omega}$ such that*

$$P_t \longrightarrow \sum_{i=1}^N a_i\, b_i\, \delta_{x_i^*}, \qquad \text{as } t \to T, \quad \text{in the weak sense of measures on } \Omega.$$

where

$$b_i := \begin{cases} 0 & \text{if } x_i^* \in \partial\Omega, \\ 1 & \text{if } x_i^* \in \Omega. \end{cases} \tag{5.3.4}$$

Démonstration. First, as a consequence of Lemma 5.3.1, the derivative in time of $\langle P_t, \varphi\rangle$ is bounded for all fixed choice of $\varphi \in \mathcal{C}^\infty$, compactly supported in Ω. This implies that $\langle P_t, \varphi\rangle$ converges as $t \to T$ for all φ. Therefore, there exists a distribution $P^* \in \mathcal{D}'(\Omega)$ such that

$$P_t \longrightarrow P^* \qquad \text{as } t \to T, \quad \text{in the distributional sense.}$$

On the other hand, since Ω is a bounded subset of $\mathbb{R}^2$, the vector $\big(x_i(t)\big)_{i=1}^N$ is bounded in $\overline{\Omega}^N$ as $t \to T$. Therefore, by compactness, there exists an increasing sequence of time (t_n) converging towards T and a family of N points $x_i^* \in \overline{\Omega}$ such that for all $i = 1 \ldots N$,

$$x_i(t_n) \longrightarrow x_i^*. \tag{5.3.5}$$

As a consequence of (5.3.4) and (5.3.5), since we have for any C^∞ map φ compactly supported in Ω that

$$\left\langle \sum_{i=1}^N a_i\, \delta_{x_i(t_n)}, \varphi \right\rangle = \sum_{i=1}^N a_i \varphi(x_i(t_n)) \longrightarrow \sum_{i=1}^N a_i b_i \varphi(x_i^*) = \left\langle \sum_{i=1}^N a_i\, b_i\, \delta_{x_i^*}, \varphi \right\rangle \qquad \text{as } n \to +\infty,$$

we have the following convergence in the distributional sense :

$$\sum_{i=1}^N a_i\, \delta_{x_i(t_n)} \longrightarrow \sum_{i=1}^N a_i\, b_i\, \delta_{x_i^*} \qquad \text{as } n \to +\infty.$$

By uniqueness of the limit, it is possible to identify

$$P^* = \sum_{i=1}^N a_i\, b_i\, \delta_{x_i^*}.$$

The fact that the convergence of P_t towards P^* in $\mathcal{D}'$ is actually a convergence in the weak sense of measures comes from the fact that the measure P_t is bounded (by $\sum_i |a_i|$) for all t. □

5.3.2 Convergence of the vortices

Lemma 5.3.3 (Accumulation points for the vortices)**.** *Let* $(t \mapsto x_i(t))_i$ *be a solution of the Euler point-vortex problem* (5.1.8) *in* Ω. *Consider a set of points* x_i^* *given by Lemma 5.3.2.*

Then, for all $i = 1 \ldots N$,

$$\Big\{x_0 \in \overline{\Omega}\ :\ \liminf_{t\to T} |x_i(t) - x_0| = 0\Big\} \ \subseteq\ \partial\Omega \cup \Big(\bigcup_{k=1}^N \{x_k^*\}\Big).$$

Démonstration. Let $x_0 \in \Omega$. Assume that there exists an index i and a sequence of times (t_n) such that

$$x_i(t_n) \longrightarrow x_0, \qquad \text{as } t \to T.$$

Since Ω is bounded, one can assume, up to an omitted extraction, that for all $j = 1 \ldots N$, there exists $x_j^\dagger \in \overline{\Omega}$ such that

$$x_j(t_n) \longrightarrow x_j^\dagger, \qquad \text{as } t \to T.$$

As in the proof of Lemma 5.3.2, we have that

$$\sum_{j=1}^N a_j\, \delta_{x_j(t_n)} \longrightarrow \sum_{j=1}^N a_j\, b_j'\, \delta_{x_j^\dagger} \qquad \text{as } n \to +\infty,$$

where the b_j' are defined as in (5.3.4) with the points $x_j^\dagger$.

By uniqueness of the limit, recalling that by Lemma 5.3.2, $P_t \to P^*$, we have that

$$\sum_{j=1}^N a_j\, b_j'\, \delta_{x_j^\dagger} = \sum_{j=1}^N a_j\, b_j\, \delta_{x_j^*}. \tag{5.3.6}$$

We focus on the left-hand side of this equation. We want to prove that the coefficient of δ_{x_0} is not 0. Let $J = \{j = 1 \dots N : x_j^\dagger = x_0\}$. Then, since $x_0 \in \Omega$, this coefficient is

$$\sum_{j\in J} a_j b_j' = \sum_{j\in J} a_j.$$

By definition of J, $i \in J$, thus J is non empty. By the non neutral cluster hypothesis (5.1.19), the coefficient of δ_{x_0} is not zero. Therefore relation (5.3.6) gives that $x_0 \in \{x_1^*, \dots, x_N^*\}$. □

Now that we have a better description of the possible accumulation points for the vortex $x_i(t)$, it remains to study the convergence of its trajectory.

Lemma 5.3.4 (The *lim inf* is actually a limit)**.** *Let* $(t \mapsto x_i(t))_i$ *be a solution of the Euler point-vortex problem* (5.1.8) *in* Ω.

If $x_0 \in \Omega$ *is such that*

$$\liminf_{t\to T^-} \left|x_i(t) - x_0\right| = 0, \tag{5.3.7}$$

then

$$\lim_{t\to T^-} \left|x_i(t) - x_0\right| = 0.$$

Démonstration. We apply Lemma 5.3.2 to construct the points $x_k^* \in \Omega$. As a consequence of Lemma 5.3.3, there exists an index k such that $x_0 = x_k^*$. Assume for the sake of contradiction that there exists $x^\dagger \neq x_0$ in $\overline{\Omega}$ such that

$$\liminf_{t\to T^-} \left|x_i(t) - x^\dagger\right| = 0. \tag{5.3.8}$$

The two *lim inf* given by (5.3.7) and by (5.3.8) imply the existence of an increasing sequence of time (t_n) converging towards T such that

$$\lim_{n\to+\infty} \left|x_i(t_{2n}) - x_k^*\right| = 0 \qquad \text{and} \qquad \lim_{n\to+\infty} \left|x_i(t_{2n+1}) - x^\dagger\right| = 0. \tag{5.3.9}$$

Let us define

$$J := \{j = 1 \dots N : x_k^* \neq x_j^*\}$$

and

$$\begin{aligned}
\delta_0 &:= \operatorname{dist}(x_k^*, \partial\Omega) > 0,\\
\delta^* &:= \min_{j\in J} |x_k^* - x_j^*| \in (0, +\infty],\\
\delta^\dagger &:= |x_k^* - x^\dagger| > 0,\\
\delta &:= \frac{1}{2}\min\{\delta_0; \delta^*; \delta^\dagger\} > 0.
\end{aligned}$$

With such definitions and since $\delta < \delta^\dagger$,

$$x^\dagger \notin \mathcal{B}(x_k^*, \delta).$$

Then, by continuity of the trajectories and using the intermediate value theorem, Condition (5.3.9) implies (when n is large enough) the existence of a time $\tau_n \in [t_{2n}; t_{2n+1}]$ such that

$$\left|x_i(\tau_n) - x_k^*\right| = \delta.$$

In other words, $x_i(\tau_n)$ belongs to the circle of center x_k^* and of radius δ, noted $\mathcal{C}(x_k^*, \delta)$. This circle being a compact set, up to an omitted extraction the following convergence holds :

$$x_i(\tau_n) \longrightarrow \widehat{x_i} \in \mathcal{C}(x_k^*, \delta).$$

This point $\widehat{x_i}$ is an accumulation point of $x_i(t)$ as $t \to T$. By definition of δ, one can easily check that $\mathcal{C}(x_k^*, \delta) \subseteq \Omega \setminus \{x_1^*, \dots, x_N^*\}$. We conclude that $\widehat{x_i}$ is an accumulation point of $x_i(t)$ as $t \to T$ which belongs to Ω but not to $\{x_1^*, \dots, x_N^*\}$. This is in contradiction with Lemma 5.3.3. □

5.3.3 Hölder regularity properties

There remain to establish the announced Hölder regularity for the trajectories that converge inside Ω as $t \to T$ and the Hölder regularity of the distance with the boundary for the vortices that collapses with $\partial\Omega$.

Choosing the right interval of time

To start with, we recall that the boundary of Ω is assumed to be smooth enough to have its curvature well-defined and bounded. This implies the existence of a $\overline{\delta} > 0$ such that for all $x \in \Omega$ such that $\text{dist}\,(x, \partial\Omega) \leq \overline{\delta}$, there exists a unique $x' \in \partial\Omega$ such that

$$|x - x'| = \text{dist}\,(x, \partial\Omega).$$

In other words, the projection of x on $\partial\Omega$ is well-defined on the set $\Omega \cap (\partial\Omega + \mathcal{B}(0, \overline{\delta}))$. This projection is denoted by $P_{\partial\Omega}$. If $\overline{\delta}$ is chosen small enough (or, which is equivalent, if we replace $\overline{\delta}$ by $\overline{\delta}/2$), then the function $P_{\partial\Omega}$ is smooth and the gradient of the distance between x and $\partial\Omega$ is given by :

$$\nabla_x \text{dist}\,(x, \partial\Omega) = \frac{x - P_{\partial\Omega} x}{|x - P_{\partial\Omega} x|}. \tag{5.3.10}$$

See for instance [30, Appendix 14.6], for details. Moreover, it satisfies the following Lipschitz estimate : for all $x, y \in \Omega \cap (\partial\Omega + \mathcal{B}(0, \overline{\delta}))$,

$$\big|P_{\partial\Omega} x - P_{\partial\Omega} y\big| \;\leq\; C_{\partial\Omega}\, |x - y|,$$

where C_Ω is a constant that depends only on the maximal curvature of $\partial\Omega$. Note that $\overline{\delta}$ also depends only of the maximal curvature of $\partial\Omega$.

Recall that we defined by (5.1.26) :

$$I := \Big\{ i = 1 \ldots N \;:\; \liminf_{t \to T^-} \;\text{dist}\,\big(x_i(t), \partial\Omega\big) = 0 \Big\}. \tag{5.3.11}$$

If $i \notin I$, by compactness, the trajectory $x_i(t)$ has an accumulation point inside Ω as $t \to T^-$. By Lemma 5.3.4 this means that

$$\forall\, i \notin I, \quad \exists\, x_i^* \in \Omega, \qquad x_i(t) \longrightarrow x_i^* \quad \text{as } t \to T^-. \tag{5.3.12}$$

If $i \in I$, then by Lemma 5.3.4, it is not possible for $x_i(t)$ to have an accumulation point inside Ω as $t \to T^-$ and thus

$$\forall i \in I, \qquad \lim_{t \to T^-} \text{dist}\,(x_i(t), \partial\Omega) = 0. \tag{5.3.13}$$

We define d_0 as

$$d_0 := \min_{i \notin I} \inf_{t \in [0,T)} \text{dist}\,(x_i(t), \partial\Omega). \tag{5.3.14}$$

Clearly $d_0 > 0$. We also define

$$\delta := \frac{1}{4} \min\{\overline{\delta}; d_0; 1\}. \tag{5.3.15}$$

From those definitions and relations (5.3.12) and (5.3.13), we get the existence of a time $T_\delta \in [0, T)$ depending on δ and thus on d_0 and Ω such that for all $t \in [T_\delta; T)$:

$$\forall\, i \notin I, \quad \text{dist}\,(x_i(t), \partial\Omega) \geq \frac{3}{4}\delta \qquad \text{and} \qquad \forall\, i \in I, \quad \text{dist}\,(x_i(t), \partial\Omega) \leq \frac{1}{4}\delta. \tag{5.3.16}$$

More precisely, we define T_δ as follows :

$$T_\delta := \inf \left\{ \tau \in [0, T) \;:\; \text{condition (5.3.16) holds true for all } t \in [\tau, T) \right\} < T. \tag{5.3.17}$$

We are going to establish the Hölder regularity properties near the collapse on the time interval $[T_\delta, T)$.

Hölder regularity property for the vortices far from the boundary

The vortices x_i with $i \notin I$ are the vortices that remain far from the boundaries of Ω. Let $i \notin I$, the equation of evolution for such a vortex is given by

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = \frac{1}{2\pi} \sum_{\substack{k \notin I \\ k \neq i}} a_k \nabla^\perp G_1\big(x_i(t) - x_k(t)\big) + f_i(t), \tag{5.3.18}$$

where f_i denotes the influence of the boundary of Ω and of the vortices that are colliding with the boundary :

$$f_i(t) = \sum_{k=1}^{N} a_k \nabla_x^\perp \gamma_\Omega\big(x_i(t), x_k(t)\big) + \frac{1}{2\pi} \sum_{k \in I} a_k \nabla^\perp G_1\big(x_i(t) - x_k(t)\big) + \sum_{m=1}^{M} c_m(t) \beta_m\big(x_i(t)\big).$$

On the interval of time $[T_\delta, T)$, it is a consequence of (5.1.6) and (5.3.16) that the function f_i is bounded by a constant that depends on a_i, on N, on Ω, on $\delta > 0$ and on ξ_m. The dependency with respect to ξ_m comes from the definition of $c_m(t)$ at (5.1.10). Therefore, the dynamics (5.3.18) satisfies (5.2.12), since for any $P \subset \{1, \dots, N\} \setminus I$,

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}t} B_P\right| &= \left|\sum_{i \in P} a_i\right|^{-1} \left| \frac{1}{2\pi} \sum_{i \in P} \sum_{\substack{k \notin I \\ k \neq i}} a_i a_k \nabla^\perp G_1\big(x_i(t) - x_k(t)\big) + \sum_{i \in I} a_i f_i(t) \right| \\
&\leq \sum_{i \in P} \sum_{k \notin P \cup I} \frac{C_0}{|x_i(t) - x_k(t)|} + C_1.
\end{aligned}$$

If we now consider $t_1 < t_2 \in [T_\delta, T)$ it is then possible to apply Lemma 5.2.7-(ii) to the dynamics (5.3.18) on the interval $[t_1, t_2)$ with $\alpha = 1$ to obtain the following Hölder estimate :

$$\forall\, t_1 < t_2 \in [T_\delta, T), \qquad \big|x_i(t_2) - x_i(t_1)\big| \leq C\sqrt{t_2 - t_1}\,.$$

The constant C depends on a_i, on N, on $\delta > 0$ and on ξ_m. To conclude the proof of Theorem 5.1.4-(i), we observe that on the time interval $[0, T_\delta]$ the trajectories are actually C^∞.

Estimate of the distances to the boundary

We now study the system of equations for the distance to the boundary of the point-vortices that collapse with the boundary : $t \mapsto \operatorname{dist}(x_i(t), \partial\Omega)$ with $i \in I$.

Lemma 5.3.5. *Consider the point-vortex dynamics $x_i(t)$ with intensities $a_i \neq 0$ on bounded multi-connected domains* (5.1.8) *and under non-neutral clusters hypothesis* (5.1.19)*. Define the set $I \subseteq \{1, \dots, N\}$ by* (5.3.11)*, the distance $\delta > 0$ by* (5.3.15) *and the time T_δ by* (5.3.17)*.*

Then, the following estimate holds for all $J \subseteq I$ non-empty :

$$\begin{aligned}
\forall\, t \in [T_\delta, T), \qquad & \left| \frac{\mathrm{d}}{\mathrm{d}t} \Big(\sum_{i \in J} a_i \Big)^{-1} \sum_{i \in J} a_i \operatorname{dist}\big(x_i(t), \partial\Omega\big) \right| \\
& \leq \sum_{i \in J} \sum_{j \in I \setminus J} \frac{C_0}{|\operatorname{dist}(x_i(t), \partial\Omega) - \operatorname{dist}(x_j(t), \partial\Omega)|} + \sum_{i \in J} \frac{C_1}{\operatorname{dist}(x_i(t), \partial\Omega)} + C_2,
\end{aligned}$$

where C_0, C_1 and C_2 are constant depending on N, a_i, Ω, ξ_m and δ.

Démonstration. Note first that $P_{\partial\Omega}$ is well-defined for $x_i(t)$ when $i \in I$ and $t \in [T_\delta, T)$ because of (5.3.16). Using (5.1.8) :

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{dist}\,(x_i(t),\partial\Omega) &= \nabla_x \mathrm{dist}\,(x_i(t),\partial\Omega)\cdot\frac{\mathrm{d}}{\mathrm{d}t}x_i(t)\\ &= \frac{x_i(t)-P_{\partial\Omega}x_i(t)}{|x_i(t)-P_{\partial\Omega}x_i(t)|}\cdot\frac{\mathrm{d}}{\mathrm{d}t}x_i(t)\\ &= g_i(t)+h_i(t)-\ell_i(t)-\frac{x_i(t)-P_{\partial\Omega}x_i(t)}{|x_i(t)-P_{\partial\Omega}x_i(t)|}\cdot\sum_{\substack{j\in I\\ j\neq i}} a_j\nabla_x^\perp\mathcal{G}_\Omega\big(x_i(t),x_j(t)\big)\end{aligned} \tag{5.3.19}$$

where

$$g_i(t) := a_i\frac{x_i(t)-P_{\partial\Omega}x_i(t)}{|x_i(t)-P_{\partial\Omega}x_i(t)|}\cdot\nabla^\perp\gamma_\Omega\big(x_i(t),x_i(t)\big),$$

$$h_i(t) := \frac{x_i(t)-P_{\partial\Omega}x_i(t)}{|x_i(t)-P_{\partial\Omega}x_i(t)|}\cdot\sum_{m=1}^{M}c_m(t)\beta_m\big(x_i(t)\big)$$

and

$$\ell_i(t) := \frac{x_i(t)-P_{\partial\Omega}x_i(t)}{|x_i(t)-P_{\partial\Omega}x_i(t)|}\cdot\sum_{j\notin I}a_j\nabla_x^\perp\mathcal{G}_\Omega\big(x_i(t),x_j(t)\big).$$

We have that the function $g_i(t)$ is bounded by a constant that depends only on Ω, see the proof of Corollary 3.6 in [20]. Concerning $h_i(t)$, the definitions of $c_m(t)$ and β_m respectively at (5.1.10) and (5.1.9) give that this term is bounded by standard elliptic estimates. The bounds depend on a_i, on N, on Ω and on ξ_m. It is a consequence of the following estimate

$$\forall x,y\in\Omega, \qquad \big|\nabla_x\mathcal{G}_\Omega(x,y)\big| \leq \frac{C_\Omega}{|x-y|}, \tag{5.3.20}$$

obtained from relations (5.1.5) and (5.1.7), that the function $t\mapsto \ell_i(t)$ is bounded by a constant that depends on a_i, Ω, N and on $\delta>0$. The remaining term in (5.3.19) is the only singular term.

Let us compute now the evolution of the barycenters for the distances to $\partial\Omega$. Let $J\subseteq I$. Recalling (5.1.5), we have that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i\in J}a_i\,\mathrm{dist}\,\big(x_i(t),\partial\Omega\big)\\ &=\sum_{i\in J}a_i\Big(g_i(t)+h_i(t)-\ell_i(t)\Big)-\sum_{i\in J}\sum_{\substack{j\in I\\ j\neq i}}a_i\,a_j\frac{x_i(t)-P_{\partial\Omega}x_i(t)}{|x_i(t)-P_{\partial\Omega}x_i(t)|}\cdot\nabla_x^\perp\mathcal{G}_\Omega\big(x_i(t),x_j(t)\big) \qquad (5.3.21)\\ &=\sum_{i\in J}a_i\,\Big(g_i(t)+h_i(t)-\ell_i(t)\Big)+Q_J(t)+R_J(t)-S_J(t)\end{aligned}$$

where

$$Q_J(t) := \frac{1}{2\pi}\sum_{i\in J}\sum_{\substack{j\in J\\ j\neq i}}a_i\,a_j\frac{x_i(t)-P_{\partial\Omega}x_i(t)}{|x_i(t)-P_{\partial\Omega}x_i(t)|}\cdot\nabla^\perp G_1\big(x_i(t)-x_j(t)\big), \tag{5.3.22}$$

$$R_J(t) := \sum_{i\in J}\sum_{\substack{j\in J\\ j\neq i}}a_i\,a_j\frac{x_i(t)-P_{\partial\Omega}x_i(t)}{|x_i(t)-P_{\partial\Omega}x_i(t)|}\cdot\nabla_x^\perp\gamma_\Omega\big(x_i(t),x_j(t)\big) \tag{5.3.23}$$

and

$$S_J(t) := \sum_{i\in J}\sum_{j\in I\setminus J}a_i\,a_j\frac{x_i(t)-P_{\partial\Omega}x_i(t)}{|x_i(t)-P_{\partial\Omega}x_i(t)|}\cdot\nabla_x^\perp\mathcal{G}_\Omega\big(x_i(t),x_j(t)\big).$$

The most singular term above is *a priori* the term Q_J in (5.3.22). Nevertheless, it can be rewritten using the symmetry property $G_1(x-y) = G_1(y-x)$ to symmetrize the double sum :

$$\begin{aligned}&\sum_{i\in J}\sum_{\substack{j\in J\\ j\neq i}} a_i\, a_j \frac{x_i(t) - P_{\partial\Omega}x_i(t)}{|x_i(t) - P_{\partial\Omega}x_i(t)|}\cdot\nabla^\perp G_1\big(x_i(t) - x_j(t)\big)\\ &= \frac{1}{2}\sum_{i\in J}\sum_{\substack{j\in J\\ j\neq i}} a_i\, a_j \left(\frac{x_i(t) - P_{\partial\Omega}x_i(t)}{|x_i(t) - P_{\partial\Omega}x_i(t)|} - \frac{x_j(t) - P_{\partial\Omega}x_j(t)}{|x_j(t) - P_{\partial\Omega}x_j(t)|}\right)\cdot\nabla^\perp G_1\big(x_i(t) - x_j(t)\big).\end{aligned} \tag{5.3.24}$$

Since we only focus on points that are close to the boundary, the gradient of $x \mapsto \mathrm{dist}\,(x, \partial\Omega)$ is a smooth function. Recalling (5.3.10) we have

$$\left|\frac{x_i(t) - P_{\partial\Omega}x_i(t)}{|x_i(t) - P_{\partial\Omega}x_i(t)|} - \frac{x_j(t) - P_{\partial\Omega}x_j(t)}{|x_j(t) - P_{\partial\Omega}x_j(t)|}\right| \le C_\Omega\big|x_i(t) - x_j(t)\big|.$$

Combining this estimate with $|\nabla G_1(x)| \le 1/|x|$ gives that the right hand side of (5.3.24) is bounded by a constant that depends only on N, a_i and Ω. Concerning R_J in (5.3.23), we make use of Property (5.1.6) on the γ_Ω function to write

$$|R_J(t)| \le C\sum_{i\in J}\frac{1}{\mathrm{dist}\,(x_i(t), \partial\Omega)}.$$

where the constant C here depends only on N, a_i and Ω. For the term $S_J(t)$, we first remark that since the distance to the boundary is smooth we have that

$$\big|\mathrm{dist}\,(x, \partial\Omega) - \mathrm{dist}\,(y, \partial\Omega)\big| \le C_\Omega\,|x-y|$$

Plugging this back into (5.3.20) leads to

$$\big|\nabla_x \mathcal{G}_\Omega(x,y)\big| \le \frac{C'_\Omega}{|\mathrm{dist}\,(x, \partial\Omega) - \mathrm{dist}\,(y, \partial\Omega)|},$$

so that,

$$|S_J(t)| \le C\sum_{i\in I}\sum_{j\in I\setminus J}\frac{1}{|\mathrm{dist}\,(x_i(t), \partial\Omega) - \mathrm{dist}\,(x_j(t), \partial\Omega)|}$$

and the constant C depends on N, a_i and Ω. Plugging all these estimates back into (5.3.21) gives the required conclusion. □

Hölder regularity property for the vortices collapsing with the boundary

With Lemma 5.3.5 at hand, it is possible to continue the proof of the Hölder regularity with arguments similar to the case of the whole plane. For that purpose, we prove :

Proposition 5.3.6 (Sufficient condition to prevent a collapse - *bis*)**.** *For $i = 1\dots N$, let $t\mapsto\zeta_i(t)$ be a family of N different points of $\mathbb{R}^p$ evolving on a time interval $[0,T)$, with $T>0$. We assume furthermore that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta_i \in L^1_{\mathrm{loc}}([0,T)).$$

Let $(a_i)_{i=1\dots N}$ satisfy (5.1.19)*. The definition of the barycenter of clusters with intensities a_i is given analogously to* (5.1.20)*.*

We assume that these points evolve such that there exists $C_0, C_1, C_2 \ge 0$ and $\alpha \ge 0$ such that

$$\forall P\in\mathcal{P}(N),\qquad \left|\frac{\mathrm{d}}{\mathrm{d}t}B_P(t)\right| \le \sum_{i\in P}\sum_{j\notin P}\frac{C_0}{|\zeta_i(t)-\zeta_j(t)|^\alpha} + \sum_{i\in P}\frac{C_1}{|\zeta_i(t)|^\alpha} + C_2. \tag{5.3.25}$$

Then there exists a constant $C_3 > 0$ such that for all $\eta \in (0,1]$, for all $t \in [0,T)$ such that

$$T - t \leq C_3\, \eta^{\alpha+1},$$

and for all indices $i \neq j \in \{1,\dots,N\}$, the two following implications are true :

$$|\zeta_i(t) - \zeta_j(t)| \geq \eta \quad \Longrightarrow \quad \forall \tau \in [t,T), \quad |\zeta_i(\tau) - \zeta_j(\tau)| \geq \frac{\eta}{2}$$

and

$$|\zeta_i(t)| \geq \eta \quad \Longrightarrow \quad \forall \tau \in [t,T), \quad |\zeta_i(\tau)| \geq \frac{\eta}{2}.$$

The constant C_3 depends only on α, a, A, C_0, C_1, C_2 and N.

The proof of this proposition is somehow similar to the proof of Proposition 5.2.5. Several extra arguments are required to take into account the additional singular term appearing at (5.3.25). The details of the new proof are delayed to Section 5.3.4.

We continue with the proof of Theorem 5.1.4.

Lemma 5.3.7. *Let $t \in [0,T) \mapsto \big(x_i(t)\big)_i$ be a solution of the point-vortex problem in a bounded domain Ω with intensities a_i satisfying the non-neutral clusters hypothesis* (5.1.19). *Recall the definition of I at* (5.3.11), *$\delta > 0$ at* (5.3.15) *and $T_\delta < T$ at* (5.3.17). *Define*

$$z_i(t) := \operatorname{dist}\big(x_i(t); \partial\Omega\big).$$

Then, there exists a constant C such that for all $i \in I$,

$$\forall\, t_1 < t_2 \in [T_\delta, T), \qquad \big|z_i(t_2) - z_i(t_1)\big| \leq\, C\, \sqrt{t_2 - t_1}\,. \tag{5.3.26}$$

Démonstration. The proof of this lemma is reminiscent of the proof of Lemma 5.2.7. Let $t_2 \in [T_\delta, T)$. For all $t \in [T_\delta, t_2)$, we define the evolution system $\zeta_k(t)$ for $k \in I \cup (N+I)$ by

$$\zeta_i(t) := z_i(t), \qquad \text{and} \qquad \zeta_{N+i}(t) := z_i(t_2), \qquad \text{where } i \in I.$$

We associate to this system some intensities b_k for $k \in I \cup (N+I)$ such that $b_i = a_i$ for $i \in I$ and such that the non-neutral clusters hypothesis (5.1.19) holds for the full family b_k. Such a choice of b_k is always possible since the non-neutral clusters hypothesis holds for the a_i. Now, let $P \subseteq I \cup (N+I)$ non empty and denote by B_P the center of vorticity associated to this dynamics. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t} B_P = \Big(\sum_{k\in P} b_k\Big)^{-1} \sum_{k \in P} b_k \frac{\mathrm{d}}{\mathrm{d}t}\zeta_k(t) = \Big(\sum_{k\in P} b_k\Big)^{-1} \sum_{i\in P\cap I} a_i \frac{\mathrm{d}}{\mathrm{d}t} z_i(t) = \Big(\sum_{k\in P} b_k\Big)^{-1} \Big(\sum_{i \in P\cap I} a_i\Big) \frac{\mathrm{d}}{\mathrm{d}t} B_{P\cap I}.$$

We now observe that it is a consequence of Lemma 5.3.5, that the dynamics $t \mapsto z_i(t)$ for $i \in I$ satisfies the bound (5.3.25) with $p = 1$ and $\alpha = 1$. This eventually implies

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t} B_P\right| &\leq \sum_{i \in P\cap I} \sum_{j \in I\setminus P} \frac{C_0'}{|z_i(t) - z_j(t)|} + \sum_{i\in P\cap I} \frac{C_1'}{|z_i(t)|} + C_2' \\ &\leq \sum_{k\in P} \sum_{\ell \in [I\cup(N+I)]\setminus P} \frac{C_0'}{|\zeta_k(t) - \zeta_\ell(t)|} + \sum_{k\in P} \frac{C_1'}{|\zeta_k(t)|} + C_2'.\end{aligned}$$

Let $i \in I$. We apply Proposition 5.3.6 with $\alpha = 1$ and $\eta := |\zeta_i(t) - \zeta_{N+i}(t)|$ to the dynamics of the ζ_k on the interval of time $[T_\delta, t_2)$. Observe that we always have $\eta \leq 1$ on the interval $[T_\delta, T)$ as a consequence of $\delta \leq 1/4$.

Since by continuity of the trajectories we have that

$$|\zeta_i(t) - \zeta_{N+i}(t)| = |z_i(t) - z_i(t_2)| \longrightarrow 0, \qquad \text{as } t \to t_2^-,$$

in view of Proposition 5.3.6, this gives that $t_2 - t > C_3\,|\zeta_i(t) - \zeta_{N+i}(t)|^2$. Indeed, otherwise it would imply that $|\zeta_i(\tau) - \zeta_{N+i}(\tau)| \geq \eta/2$ for all $\tau \in [t, t_2]$. Therefore we have that

$$|\zeta_i(t) - \zeta_{N+i}(t)| = |z_i(t) - z_i(t_2)| \leq C\,\sqrt{t_2 - t}\,.$$

The estimate above then gives (5.3.26) and concludes the proof. □

To conclude the proof of Theorem 5.1.4-(*ii*), we combine the Hölder estimate provided by Lemma 5.3.7 with the remark that on the time interval $[0, T_\delta]$ the trajectories are C^∞. One can directly check that the dependency of the Hölder constant with respect to the initial datum actually reduces to a dependency with respect to d_0 defined at (5.3.14).

To conclude the proof of Theorem 5.1.4, there only remains to establish Proposition 5.3.6.

5.3.4 Proof of Proposition 5.3.6

We first prove a preliminary Lemma.

Lemma 5.3.8 (Cluster close to zero). *Let $\mathfrak{P}$ be any partition of $\{1, \dots, N\}$ satisfying the existence of $\kappa \in (0, \frac{1}{4})$, $\delta > 0$ and points $\zeta_i \in \mathbb{R}^p$ such that*

$$\forall\, P \in \mathfrak{P}, \quad \forall\, k, \ell \in P, \qquad |\zeta_k - \zeta_\ell| \leq \delta \tag{5.3.27}$$

and

$$\forall P \neq P' \in \mathfrak{P}, \quad \forall k \in P, \quad \forall \ell \in P', \quad |\zeta_k - \zeta_\ell| \geq \kappa^{-1}\delta. \tag{5.3.28}$$

Then, there exists a unique set $\widehat{P} \in \mathfrak{P} \cup \{\emptyset\}$ satisfying the following properties.

(i) The following relation holds :

$$\forall k \notin \widehat{P}, \quad |\zeta_k| \geq \kappa^{-1}\delta/4$$

(ii) If $\widehat{P} \neq \emptyset$, the following relation holds :

$$\exists \ell \in \widehat{P}, \quad |\zeta_\ell| < \kappa^{-1}\delta/4.$$

(iii) The following relation holds :

$$\forall j \in \widehat{P}, \quad |\zeta_j| < (\kappa^{-1}/4 + 1)\delta.$$

Démonstration. If there exists $\ell \in \{1, \dots, N\}$ such that $|\zeta_\ell| < \kappa^{-1}\delta/4$, then we denote by $\widehat{P}$ the cluster containing ℓ in the partition $\mathfrak{P}$, else we set $\widehat{P} = \emptyset$. If $\widehat{P} = \emptyset$, that means that (*i*) must be satisfied. Conditions (*ii*) and (*iii*) are true since they are empty. If $\widehat{P} \neq \emptyset$, by construction, (*ii*) is satisfied and gives (*iii*) by (5.3.27). Using relation (5.3.28), we have for every $k \notin \widehat{P}$,

$$|\zeta_k - \zeta_\ell| \geq \kappa^{-1}\delta.$$

Therefore

$$|\zeta_k| = |\zeta_k - \zeta_\ell| - |\zeta_\ell| \geq \kappa^{-1}\delta - \kappa^{-1}\delta/4 \geq \kappa^{-1}\delta/4.$$

Thus (*i*) is satisfied. It is clear that (*i*) and (*ii*) define a unique set $\widehat{P} \in \mathfrak{P} \cup \{\emptyset\}$. □

We continue with the proof of Proposition 5.3.6. We proceed as in the proof of Proposition 5.2.5. We can assume without loss of generality that $\max\{C_0, C_1, C_2\} > 0$. We fix once and for all some $\eta \in (0, 1]$. Let t_1 such that

$$T - t_1 \leq C_3\,\eta^{\alpha+1}.$$

for some constant C_3 to be chosen later. During the proof, we will impose several conditions on the constant C_3 and at the end of the proof we will observe that all these conditions can be satisfied for a constant C_3 which is independent of η.

Let $0 < \kappa < \frac{A}{16a}$. Recall that A and a are respectively defined by relations (5.2.1) and (5.2.2) and that $A \neq 0$ by hypothesis (5.1.19). Remark that $\kappa \leq 1/16$ since $A \leq a$.

We start by constructing partitions of $\{1, \dots, N\}$ with an iterative process. We first invoke the corollary of the balls lemma (Corollary 5.2.4) to the points $\zeta_k(t_1)$ to build the first partition $\mathfrak{P}^1$ by choosing $d := \frac{1}{8}\kappa\eta$. This gives the first partition $\mathfrak{P}^1$ and a real number δ_1 satisfying

$$\frac{1}{8}\left(\frac{\kappa}{8}\right)^N \kappa\eta \leq \delta_1 < \frac{1}{8}\kappa\eta \tag{5.3.29}$$

such that

$$\forall\, P \in \mathfrak{P}^1, \quad \forall\, k, \ell \in P, \qquad |\zeta_k(t_1) - \zeta_\ell(t_1)| \leq \delta_1$$

and

$$\forall P \neq P' \in \mathfrak{P}^1, \quad \forall k \in P, \quad \forall \ell \in P', \quad |\zeta_k(t_1) - \zeta_\ell(t_1)| \geq \kappa^{-1}\delta_1.$$

We now define

$$r := \min\left\{\frac{1}{8}\,;\, \frac{A}{8a\kappa} - 2\right\} > 0,$$

and

$$s := r\left(\frac{\kappa}{8}\right)^N.$$

Note that $r \in (0, \frac{1}{8})$.

We now build iteratively a decreasing sequence of positive numbers δ_q, an increasing finite sequence of times (t_q) and a finite number of partitions $\mathfrak{P}^q$ satisfying

$$\forall\, P \in \mathfrak{P}^q, \quad \forall\, k, \ell \in P, \qquad |\zeta_k(t_q) - \zeta_\ell(t_q)| \leq \delta_q \tag{I}$$

and

$$\forall P \neq P' \in \mathfrak{P}^q, \quad \forall k \in P, \quad \forall \ell \in P', \quad |\zeta_k(t_q) - \zeta_\ell(t_q)| \geq \kappa^{-1}\delta_q. \tag{II}$$

Assuming that $\mathfrak{P}^q$ is constructed with those properties we can define $\widehat{P}_q \in \mathfrak{P}^q \cup \{\emptyset\}$ by applying Lemma 5.3.8 to $\mathfrak{P}^q$. This means that $\widehat{P}_q$ satisfies

$$\begin{cases} \forall k \notin \widehat{P}_q, & |\zeta_k(t_q)| \geq \kappa^{-1}\delta_q/4 \\ \forall j \in \widehat{P}_q, & |\zeta_j(t_q)| < (\kappa^{-1}/4 + 1)\delta_q. \end{cases} \tag{5.3.30}$$

The construction proceeds as follows. If the following relations are satisfied

$$\begin{cases} \forall\, k \notin \widehat{P}_q, \quad \forall \tau \in [t_q, T), \qquad |\zeta_k(\tau) - \zeta_k(t_q)| \leq \kappa^{-1}\delta_q/8 \\ \forall k \in \widehat{P}_q, \quad \forall \tau \in [t_q, T), \qquad |\zeta_k(\tau)| \leq \dfrac{3}{8}\kappa^{-1}\delta_q, \end{cases} \tag{$\star$}$$

then the construction stops. Else, we will construct a time t_{q+1} satisfying

$$\begin{cases} \forall\, k \notin \widehat{P}_q, \quad \forall \tau \in [t_q, t_{q+1}], \qquad |\zeta_k(\tau) - \zeta_k(t_q)| \leq \kappa^{-1}\delta_q/8 \\ \forall\, k \in \widehat{P}_q, \quad \forall \tau \in [t_q, t_{q+1}], \qquad |\zeta_k(\tau)| \leq \dfrac{3}{8}\kappa^{-1}\delta_q, \end{cases} \tag{III}$$

a real number δ_{q+1} satisfying

$$s\delta_q \leq \delta_{q+1} < r\delta_q \tag{IV}$$

and the next partition $\mathfrak{P}^{q+1}$ satisfying (I) and (II) at step $q+1$ as well as

$$\mathfrak{P}^{q+1} \text{ is a sub-partition of } \mathfrak{P}^{q+1} \tag{V}$$

and

$$\mathfrak{P}^{q+1} \neq \mathfrak{P}^{q} \quad \text{or} \quad \big(\widehat{P}_q \neq \emptyset \text{ and } \widehat{P}_{q+1} = \emptyset\big). \tag{VI}$$

Let us observe that condition (VI) ensures that the construction has a finite number of steps since we will prove that necessarily, $\widehat{P}_{q+1} \subset \widehat{P}_q$ and thus if this set is empty at one step, it will remain empty afterwards. Therefore $\mathfrak{P}^{q+1}$ is a strict sub-partition of $\mathfrak{P}^q$ except for at most one step.

Let $q \in \mathbb{N}^*$ be fixed and assume that the partitions $\mathfrak{P}^{q'}$ are constructed for all $q' = 1 \dots q$. Assume that $(\star)$ is not satisfied. We now construct t_{q+1}, δ_{q+1} and $\mathfrak{P}^{q+1}$.

Since $(\star)$ is not satisfied, we can define the time $t_{q+1} \in [t_q, T)$ as being the largest time such that

$$\forall\, l \notin \widehat{P}_q, \quad \forall\, \tau \in [t_q, t_{q+1}], \qquad |\zeta_\ell(\tau) - \zeta_\ell(t_q)| \leq \kappa^{-1}\delta_q/8$$

and

$$\forall\, k \in \widehat{P}_q, \quad \forall\, \tau \in [t_q, t_{q+1}], \qquad |\zeta_k(\tau)| \leq \frac{3}{8}\kappa^{-1}\delta_q.$$

By continuity of the trajectories, such a time t_{q+1} does exist. This definition ensures that (III) holds true.

To define the new partition $\mathfrak{P}^{q+1}$, we apply Corollary 5.2.4 to the points $(\zeta_j(t_{q+1}))_j$ with $d = r\delta_q$. This gives the partition $\mathfrak{P}^{q+1}$ and a real number $\delta_{q+1} > 0$ such that (I), (II) hold true at step $q+1$ and (IV) holds true.

We now prove (V). For any $m \in P, n \in P'$, with $P \neq P' \in \mathfrak{P}^q \setminus \{\widehat{P}_q\}$ we have that

$$\begin{aligned}|\zeta_m(t_{q+1}) - \zeta_n(t_{q+1})| &= |\zeta_m(t_q) - \zeta_n(t_q) + \zeta_m(t_{q+1}) - \zeta_m(t_q) + \zeta_n(t_q) - \zeta_n(t_{q+1})| \\ &\geq |\zeta_m(t_q) - \zeta_n(t_q)| - |\zeta_m(t_{q+1}) - \zeta_m(t_q)| - |\zeta_n(t_q) - \zeta_n(t_{q+1})|.\end{aligned}$$

We bound the first term using Hypothesis (II) and the other two terms by using Hypothesis (III) to obtain that

$$|\zeta_m(t_{q+1}) - \zeta_n(t_{q+1})| \geq \kappa^{-1}\delta_q - 2\kappa^{-1}\delta_q/8 \geq \kappa^{-1}\delta_q/2.$$

By Hypothesis (IV), we know that $\delta_{q+1} < r\delta_q$. Since $r < 1/8$ and $16 \leq \kappa^{-1}$, we infer that

$$\kappa^{-1}\delta_q > 128\,\delta_{q+1} \quad \text{ and } \quad \kappa^{-1}\delta_q > 16(\kappa^{-1}/4 + 1)\delta_{q+1} \tag{5.3.31}$$

Consequently, we have that

$$|\zeta_m(t_{q+1}) - \zeta_n(t_{q+1})| > \delta_{q+1}.$$

In light of Hypothesis (I) at step $q+1$, this proves that m and n do not belong to the same cluster in the partition $\mathfrak{P}^{q+1}$.

Now if $m \in P \in \mathfrak{P}^q \setminus \{\widehat{P}_q\}$ and $n \in \widehat{P}_q$, then

$$\begin{aligned}|\zeta_m(t_{q+1}) - \zeta_n(t_{q+1})| &= |\zeta_m(t_{q+1}) - \zeta_m(t_q) + \zeta_m(t_q) - \zeta_n(t_q) + \zeta_n(t_q) - \zeta_n(t_{q+1})| \\ &\geq |\zeta_m(t_q) - \zeta_n(t_q)| - |\zeta_m(t_{q+1}) - \zeta_m(t_q)| - |\zeta_n(t_q)| - |\zeta_n(t_{q+1})|.\end{aligned}$$

We bound the first term using Hypothesis (II) and the other three using Hypothesis (III) to obtain that

$$|\zeta_m(t_{q+1}) - \zeta_n(t_{q+1})| \geq \kappa^{-1}\delta_q - \kappa^{-1}\delta_q/8 - \frac{3}{8}\kappa^{-1}\delta_q - \frac{3}{8}\kappa^{-1}\delta_q = \kappa^{-1}\delta_q/8.$$

Recalling relation (5.3.31), this gives that

$$|\zeta_m(t_{q+1}) - \zeta_n(t_{q+1})| > \delta_{q+1}.$$

In conclusion, if m and n do not belong to the same cluster in $\mathfrak{P}^q$, they do not belong to the same cluster in $\mathfrak{P}^{q+1}$. This proves that $\mathfrak{P}^{q+1}$ is a sub-partition of $\mathfrak{P}^q$. Hence (V) is proved.

There only remains to prove (VI). We start by proving an inclusion property. Since $\mathfrak{P}^{q+1}$ satisfies (I) and (II), we can already define $\widehat{P}_{q+1}$. Let us prove that

$$\widehat{P}_{q+1} \subset \widehat{P}_q. \tag{5.3.32}$$

Indeed, $\forall k \notin \widehat{P}_q$, by (5.3.30) and Hypothesis (III) we have that

$$|\zeta_k(t_{q+1})| \geq |\zeta_k(t_q)| - |\zeta_k(t_q) - \zeta_k(t_{q+1})| \geq \kappa^{-1}\delta_q(1/4 - 1/8) = \kappa^{-1}\delta_q/8.$$

Recalling once again relation (5.3.31), we have that

$$|\zeta_k(t_{q+1})| > (\kappa^{-1}/4 + 1)\delta_{q+1}.$$

Relations (5.3.30) at step $q+1$ conclude that $k \notin \widehat{P}_{q+1}$. Therefore $\widehat{P}_{q+1} \subset \widehat{P}_q$.

It is not necessarily true that $\mathfrak{P}^{q+1} \neq \mathfrak{P}^q$ for all q. In order to prove (VI), we need to separate the analysis into two cases. Since t_{q+1} is the largest time such that Hypothesis (III) hold true, we either have that

$$\exists\, k \in \widehat{P}_q, \qquad |\zeta_k(t_{q+1})| = \frac{3}{8}\kappa^{-1}\delta_q, \tag{5.3.33}$$

we denote this Case 1, or that

$$\exists\, k \notin \widehat{P}_q, \qquad |\zeta_k(t_{q+1})) - \zeta_k(t_q)| = \kappa^{-1}\delta_q/8. \tag{5.3.34}$$

This is Case 2.

• **Case 1 :** Assume that (5.3.33) holds true for a given $k \in \widehat{P}_q$. In particular, this requires that $\widehat{P}_q \neq \emptyset$.

If $\mathfrak{P}^{q+1} \neq \mathfrak{P}^q$, then (VI) is proved. Recall that (I), (II), (III), (IV) and (V) are proved. We have thus correctly constructed $\mathfrak{P}^{q+1}$ and end Case 1 here.

Assume now that $\mathfrak{P}^{q+1} = \mathfrak{P}^q$. In this case we need to prove that $\widehat{P}_{q+1} = \emptyset$.

Let $\ell \in \widehat{P}_q$. Since $\widehat{P}_q \in \mathfrak{P}^q$, we have in particular that $\widehat{P}_q \in \mathfrak{P}^{q+1}$. Consequently, by condition (I) used at step $q+1$,

$$|\zeta_k(t_{q+1}) - \zeta_\ell(t_{q+1})| \leq \delta_{q+1}.$$

Combining this with the fact that k satisfies (5.3.33) gives that

$$\begin{aligned} |\zeta_\ell(t_{q+1})| &= |\zeta_\ell(t_{q+1}) - \zeta_k(t_{q+1}) + \zeta_k(t_{q+1})| \\ &\geq |\zeta_k(t_{q+1})| - |\zeta_\ell(t_{q+1}) - \zeta_k(t_{q+1})| \\ &\geq \frac{3}{8}\kappa^{-1}\delta_q - \delta_{q+1}. \end{aligned}$$

Recalling relation (5.3.31), we observe that

$$\frac{3}{8}\kappa^{-1}\delta_q - \delta_{q+1} \geq (\kappa^{-1}/4 + 1)\delta_{q+1}.$$

Therefore,

$$|\zeta_\ell(t_{q+1})| \geq (\kappa^{-1}/4 + 1)\delta_{q+1}.$$

This relation combined with relation (5.3.30) at step $q+1$ implies that $\ell \notin \widehat{P}_{q+1}$. Therefore any $l \in \widehat{P}_q$ satisfies $\ell \notin \widehat{P}_{q+1}$. By relation (5.3.32) this means that $\widehat{P}_{q+1} = \emptyset$. Therefore, (VI) is proved. This concludes Case 1.

• **Case 2 :** Assume that (5.3.34) holds true for a given $k \notin \widehat{P}_q$. Let us prove that $\mathfrak{P}^{q+1} \neq \mathfrak{P}^q$.

Let $P \in \mathfrak{P}^q$ such that $k \in P$. For $\tau \in [t_q, t_{q+1}]$ and $j \in \widehat{P}_q$, by Hypothesis (III) we have that

$$|\zeta_j(\tau) - \zeta_j(t_q)| \leq |\zeta_j(\tau)| + |\zeta_j(t_q)| \leq \frac{6}{8}\kappa^{-1}\delta_q. \tag{5.3.35}$$

This last relation is also true for $j \notin \widehat{P}_q$ because of the stronger relation (III). Then, for any $i \in P$, any $j \notin P$ and any $\tau \in [t_q, t_{q+1}]$ we have that

$$|\zeta_i(\tau) - \zeta_j(\tau)| \geq |\zeta_i(t_q) - \zeta_j(t_q)| - |\zeta_i(\tau) - \zeta_i(t_q)| - |\zeta_j(\tau) - \zeta_j(t_q)|.$$

We bound the first term using Hypothesis (II), then recalling that $P \neq \widehat{P}_q$, we bound the second term using Hypothesis (III) and the third with relation (5.3.35) to obtain that

$$|\zeta_i(\tau) - \zeta_j(\tau)| \geq \kappa^{-1}\delta_q - \frac{1}{8}\kappa^{-1}\delta_q - \frac{6}{8}\kappa^{-1}\delta_q = \kappa^{-1}\delta_q/8. \tag{5.3.36}$$

Relation (5.3.30) and Hypothesis (III) give that for every $\tau \in [t_q, t_{q+1}]$ and for every $i \in P$,

$$|\zeta_i(\tau)| \geq |\zeta_i(t_q)| - |\zeta_i(t_q) - \zeta_i(\tau)| \geq \kappa^{-1}\delta_q/4 - \kappa^{-1}\delta_q/8 = \kappa^{-1}\delta_q/8. \tag{5.3.37}$$

Recalling relation (5.3.25) we have that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}B_P(\tau)\right| \leq \sum_{i\in P}\sum_{j\notin P}\frac{C_0}{|\zeta_i(\tau) - \zeta_j(\tau)|^\alpha} + \sum_{i\in P}\frac{C_1}{|\zeta_i(\tau)|^\alpha} + C_2.$$

Plugging relations (5.3.36) and (5.3.37) into this last relation gives

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}B_P(\tau)\right| &\leq \sum_{i\in P}\sum_{j\notin P}\frac{C_0}{(\kappa^{-1}\delta_q/8)^\alpha} + \sum_{i\in P}\frac{C_1}{(\kappa^{-1}\delta_q/8)^\alpha} + C_2 \\ &\leq \frac{8^\alpha N^2(C_0 + C_1)}{(\kappa^{-1}\delta_q)^\alpha} + C_2.\end{aligned}$$

Up to a multiplicative constant, this estimate is identical to the estimate (5.2.22) obtained in the proof of Proposition 5.2.5. Provided that the constant C_3 is small enough, the same argument as in page 119 leads to $\mathfrak{P}^{q+1} \neq \mathfrak{P}^q$. The hypothesis that is required on C_3 in this case, which plays the role of relation (5.2.24), is :

$$C_3\eta^{\alpha+1} \leq \frac{a}{A}\delta_q\left(\frac{8^\alpha\, N^2\,(C_0 + C_1)}{(\kappa^{-1}\delta_q)^\alpha} + C_2\right)^{-1}. \tag{5.3.38}$$

Condition (VI) is proved. This concludes Case 2 and our construction.

It is clear that the situation

$$\widehat{P}_q \neq \emptyset \text{ and } \widehat{P}_{q+1} = \emptyset$$

can happen at most one time since we have proved relation (5.3.32) holds at every step of the construction. Therefore this iterative process has at most N steps, namely $q = 1 \ldots Q$ with $Q \leq N + 1$. We recall that Hypothesis $(\star)$ must hold true at the final step Q.

Recalling that δ_1 satisfies (5.3.29), we can prove by induction, using the construction condition (IV) and the fact that $s < \left(\frac{\kappa}{8}\right)^N$, that

$$\frac{1}{8}s^q\kappa\eta \leq \delta_q \leq r^{q-1}\kappa\eta\frac{1}{8}. \tag{5.3.39}$$

We now conclude the proof of Proposition 5.3.6. We first establish the following intermediate property : for every $i \in \{1, \ldots, N\}$ and for every $\tau \in [t_1, T)$,

$$|\zeta_i(\tau) - \zeta_i(t_1)| \leq \eta/4. \tag{5.3.40}$$

Indeed, in the case where $i \notin \widehat{P}_1$, we first observe that $\forall q \in \{1,\dots,Q\}$, $i \notin \widehat{P}_q$. If we set the convention that $t_{Q+1} = T$, then there exists a unique $q \in \{1,\dots,Q\}$ such that $\tau \in [t_q, t_{q+1})$. We write

$$|\zeta_i(\tau) - \zeta_i(t_1)| \le \sum_{q'=1}^{q-1} |\zeta_i(t_{q'+1}) - \zeta_i(t_{q'})| + |\zeta_i(\tau) - \zeta_i(t_q)|.$$

Then, by construction hypothesis (III) and $(\star)$ we have that

$$|\zeta_i(\tau) - \zeta_i(t_1)| \le \sum_{q'=1}^{q} \kappa^{-1}\delta_{q'}/8.$$

By relation (5.3.39) and the fact that $r < 1/8$ this yields

$$|\zeta_i(\tau) - \zeta_i(t_1)| \le \frac{1}{8}\sum_{q'=1}^{q} r^{q'-1}\eta/8 \le \eta/4.$$

If on the contrary $i \in \widehat{P}_1$, then we define $q \in \{1,\dots,Q\}$ as being the greatest index such that $i \in \widehat{P}_q$. We have by construction hypothesis (III) and $(\star)$ and the fact that the sequence δ_q is decreasing that for any $\tau \in [t_1, t_{q+1})$

$$|\zeta_i(\tau) - \zeta_i(t_1)| \le |\zeta_i(\tau)| + |\zeta_i(t_1)| \le 2\frac{3}{8}\kappa^{-1}\delta_1 \le \eta/8.$$

Moreover by Hypothesis (III), $(\star)$ and relation (5.3.39) we have that for $\tau \ge t_{q+1}$ that

$$|\zeta_i(\tau) - \zeta_i(t_{q+1})| \le \sum_{q'=q+1}^{\infty} \kappa^{-1}\delta_{q'}/8 \le \eta/8 \tag{5.3.41}$$

so that

$$|\zeta_i(\tau) - \zeta_i(t_1)| \le |\zeta_i(\tau) - \zeta_i(t_{q+1})| + |\zeta_i(t_{q+1}) - \zeta_i(t_1)| \le \eta/4. \tag{5.3.42}$$

We have proved relation (5.3.40).

Let i and j such that $|\zeta_i(t_1) - \zeta_j(t_1)| \ge \eta$. Relation (5.3.40) applied on i and j gives that

$$\begin{aligned}|\zeta_i(\tau) - \zeta_j(\tau)| &= |\zeta_i(\tau) - \zeta_i(t_1) + \zeta_i(t_1) - \zeta_j(t_1) + \zeta_j(t_1) - \zeta_j(\tau)|\\ &\ge |\zeta_i(t_1) - \zeta_j(t_1)| - |\zeta_i(\tau) - \zeta_i(t_1)| - |\zeta_j(t_1) - \zeta_j(\tau)|\\ &\ge \eta - 2\eta/4 = \eta/2.\end{aligned}$$

This proves relation (5.3.41) of Proposition 5.3.6.

Now if i is such that $|\zeta_i(t_1)| \ge \eta$ then for every $\tau \in [t_1, T)$,

$$|\zeta_i(\tau)| \ge |\zeta_i(t_1)| - |\zeta_i(t_1) - \zeta_i(\tau)| \ge \eta/2,$$

as a consequence of relation (5.3.40). This proves (5.3.42).

What remains to prove is the fact that C_3 does not depend on η. Proceeding as in the proof of Proposition 5.2.5, starting page (5.2.28), we can take

$$C_3 = \frac{a}{A} s^{(N+1)(\alpha+1)} \frac{\kappa}{8^\alpha N^2 (C_0 + C_1) + C_2}$$

to ensure that relation (5.3.38) hold for every $q \in \{1,\dots,Q\}$. Choosing κ and replacing s by its value gives the announced constant C_3. □

Appendix

5.4 Optimality of the Hölder exponent and self-similar collapses

This appendix is devoted to the proof of the existence of collapses for all values of $\alpha > 0$ and to check that the Hölder exponent given by Theorem 5.1.3 is optimal. The first part of this appendix studies necessary conditions to have a self-similar collapse. In the second part, we exhibit an example of self-similar collapse in the case of the 3-vortex problem for any value of $\alpha > 0$.

We say that a solution is a self-similar collapse at time $T > 0$ if there exist C^1 maps $f : [0,T] \to \mathbb{R}_+$ and $\theta : [0,T) \to \mathbb{R}$ satisfying

$$\begin{cases} f(0) = 1 \\ f(T) = 0 \\ \theta(0) = 0 \end{cases} \tag{5.4.1}$$

and for every $j \in \{1,\dots,N\}$, and every $t \in [0,T)$

$$x_j(t) = f(t)\, x_j(0)\, e^{\mathbf{i}\theta(t)} \tag{5.4.2}$$

where $\mathbf{i}$ is the complex unit ($\mathbf{i}^2 = -1$). Recall that taking the "$\perp$", the rotation of angle $\pi/2$ in $\mathbb{R}^2$, is equivalent to the multiplication by $\mathbf{i}$ in $\mathbb{C}$. By continuity of f, the condition $f(T) = 0$ implies that for every $j \in \{1,\dots,N\}$,

$$x_j(t) \longrightarrow 0, \qquad \text{as } t \to T^-. \tag{5.4.3}$$

Moreover, a necessary condition for the collapse to happen exactly at time $T > 0$ is to have $f(t) > 0$ for every $t < T$.

5.4.1 Necessary conditions for a self-similar collapse

We consider the α-point-vortex dynamic (5.1.14). The case $\alpha = 1$ (corresponding to Euler point-vortices in the plane) being already well-known [2, 4, 37, 44, 54], we fully concentrate here on the case $\alpha \neq 1$. The aim is to extract some necessary conditions on the intensities a_i and on the starting positions $x_1(0),\dots,x_N(0)$ of a configuration to have a self similar collapse.

We introduce $l_{ij} = |x_i - x_j|$. Since the Hamiltonian (5.1.16) is preserved during the motion, the first condition we obtain is that $H(t) = H(0)$ holds at all times. However relation (5.4.2) gives that

$$H(t) = \frac{1}{f(t)^{\alpha-1}} H(0).$$

Therefore the Hamiltonian must be equal to 0, which reads

$$\sum_{i\neq j} \frac{a_i a_j}{l_{ij}^{\alpha-1}(0)} = 0. \tag{5.4.4}$$

The second invariant we have is

$$L(t) = \sum_{i\neq j} a_i a_j l_{ij}^2(t).$$

Indeed, $L = \sum_{i\neq j} a_i a_j |x_i - x_j|^2 = 2\left(\sum_{i=1}^N a_i\right) I - 2|M|^2$ with M defined by (5.1.17) and I defined by (5.1.18). Both M and I are constant in time so L is also constant in time. Relation (5.4.3) gives that $l_{ij}(t)$ tends to 0 as $t \to T^-$. This implies that $L(0) = 0$, which reads

$$\sum_{i\neq j} a_i a_j l_{ij}^2(0) = 0. \tag{5.4.5}$$

Relations (5.4.4) and (5.4.5) are two necessary conditions on the intensities and the starting positions for a self similar collapse to occur. These conditions are the generalization for all α of the already known conditions when $\alpha = 1$ (see [4]) and when $\alpha = 2$ (see [72]).

We now go further to find the necessary expression of f appearing in (5.4.2). In the case of self-similar collapses, using relation (5.4.2), we compute the evolution l_{ij}^2 for α models (5.1.14) :

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} l_{ij}^2 &= 2(x_i - x_j) \cdot \frac{\mathrm{d}}{\mathrm{d}t}(x_i - x_j) \\
&= 2(x_i - x_j) \cdot \left(\sum_{k \neq i} a_k \frac{(x_i - x_k)^\perp}{|x_i - x_k|^{\alpha+1}} - \sum_{\ell \neq j} a_\ell \frac{(x_j - x_\ell)^\perp}{|x_j - x_\ell|^{\alpha+1}} \right) \\
&= 2 \left[(x_i(0) - x_j(0)) f(t) e^{\mathbf{i}\theta} \right] \cdot \left[\frac{e^{\mathbf{i}\theta}}{f(t)^\alpha} \left(\sum_{k \neq i} a_k \frac{(x_i(0) - x_k(0))^\perp}{|x_i(0) - x_k(0)|^{\alpha+1}} - \sum_{\ell \neq j} a_\ell \frac{(x_j(0) - x_\ell(0))^\perp}{|x_j(0) - x_\ell(0)|^{\alpha+1}} \right) \right] \\
&= \frac{2}{f(t)^{\alpha-1}} (x_i(0) - x_j(0)) \cdot \left(\sum_{k \neq i} a_k \frac{(x_i(0) - x_k(0))^\perp}{|x_i(0) - x_k(0)|^{\alpha+1}} - \sum_{\ell \neq j} a_\ell \frac{(x_j(0) - x_\ell(0))^\perp}{|x_j(0) - x_\ell(0)|^{\alpha+1}} \right)
\end{aligned}
$$

Therefore, there exist constants $C_{i,j}$ independent of the time such that

$$
f'(t) f(t) = \frac{C_{i,j}}{f(t)^{\alpha-1}}, \tag{5.4.6}
$$

where

$$
C_{i,j} := \frac{x_i(0) - x_j(0)}{|x_i(0) - x_j(0)|^2} \cdot \left(\sum_{k \neq i} a_k \frac{(x_i(0) - x_k(0))^\perp}{|x_i(0) - x_k(0)|^{\alpha+1}} - \sum_{\ell \neq j} a_\ell \frac{(x_j(0) - x_\ell(0))^\perp}{|x_j(0) - x_\ell(0)|^{\alpha+1}} \right).
$$

We now observe that the equality (5.4.6) implies that the constants $C_{i,j}$ do not actually depend on i and j. We denote $C = C_{i,j}$ and solve (5.4.6) to obtain

$$
f(T)^{\alpha+1} - f(t)^{\alpha+1} = (\alpha + 1) C (T - t).
$$

Recalling from (5.4.1) that $f(T) = 0$, we observe that we must have that $C < 0$. This gives

$$
f(t) = C' (T - t)^{\frac{1}{\alpha+1}}
$$

with

$$
C' = (-C(\alpha + 1))^{\frac{1}{\alpha+1}}.
$$

Recalling from (5.4.1) that $f(0) = 1$, we have $C' = \frac{1}{T^{\frac{1}{\alpha+1}}}$, so that we finally obtain that the function f is necessarily equal to

$$
f(t) = \left(\frac{T - t}{T} \right)^{\frac{1}{\alpha+1}}. \tag{5.4.7}
$$

We do the same for the function θ. Taking the derivative in time of relation (5.4.2) gives

$$
\sum_{k \neq j} \mathbf{i} a_k \frac{x_j - x_k}{|x_j - x_k|^{\alpha+1}} = f'(t) x_j(0) e^{\mathbf{i}\theta(t)} + f(t) x_j(0) \mathbf{i} \theta'(t) e^{\mathbf{i}\theta(t)}.
$$

Thus

$$
\frac{1}{f(t)^\alpha} \sum_{k \neq j} a_k \frac{x_j(0) - x_k(0)}{|x_j(0) - x_k(0)|^{\alpha+1}} = -\mathbf{i} f'(t) x_j(0) + f(t) x_j(0) \theta'(t).
$$

Recalling relations (5.4.6) and (5.4.7) we have that

$$-\mathbf{i}Cx_j(0) + \frac{T-t}{T}\theta'(t)x_j(0) = D_j, \tag{5.4.8}$$

where

$$D_j = \sum_{k\neq j} a_k \frac{x_j(0) - x_k(0)}{|x_j(0) - x_k(0)|^{\alpha+1}}.$$

We remark that relation (5.4.8) implies that the quantity $\frac{T-t}{T}\theta'(t)$ is constant in time. We denote its value by $D \in \mathbb{R}$. Recalling from (5.4.1) that $\theta(0) = 0$ we infer that

$$\theta(t) = -DT \ln \frac{T-t}{T}. \tag{5.4.9}$$

Gathering relations (5.4.2), (5.4.7) and (5.4.9) we obtain that the trajectories of a self-similar collapse are necessary of the following form :

$$x_j(t) = x_j(0) \left(\frac{T-t}{T}\right)^{\frac{1}{\alpha+1}} \exp\left(-\mathbf{i}DT \ln \frac{T-t}{T}\right).$$

It is a direct computation to check that in the expression above, the function $x_j(t)$ belongs to the Hölder space $\mathcal{C}^{0,\beta}([0,T];\mathbb{C})$ if and only if $\beta \leq \frac{1}{\alpha+1}$.

5.4.2 Existence of a self-similar collapse

We now prove that, for any $\alpha > 0$, there exists an initial configuration leading to a self-similar collapse for the point-vortex dynamic (5.1.14).

We construct our example in the case $N = 3$, $a_2 = a_3 = 1$ and $a_1 = a \in \mathbb{R}^*$ to be fixed later. We define $A = |x_2 - x_3|$, $B = |x_3 - x_1|$ and $C = |x_1 - x_2|$. Let $\lambda \in (0,1)$. We choose values for $x_1(0)$, $x_2(0)$ and $x_3(0)$ such that these three points form a direct orthogonal triangle with $A(0) = 1$, $B(0) = \lambda$ and $C(0) = \sqrt{\lambda^2 + 1}$. We now want to find a value for λ and a such that the associated solution is a self-similar collapse. More precisely, we are going to exhibit a value for a and λ such that $B(t) = \lambda A(t)$ and $C(t) = \sqrt{\lambda^2+1}A(t)$ hold for any time t, and such that $A(t) \to 0$ as $t \to T$ for some $T > 0$.

It is a direct computation using the point-vortex equations (5.1.14) to check that (see also [72]),

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} - \frac{\mathrm{d}x_3}{\mathrm{d}t} = a\frac{(x_2-x_1)^\perp}{C^{\alpha+1}} + \frac{(x_2-x_3)^\perp}{A^{\alpha+1}} - a\frac{(x_3-x_1)^\perp}{B^{\alpha+1}} - \frac{(x_3-x_2)^\perp}{A^{\alpha+1}}$$

so that

$$(x_2 - x_3)\cdot\frac{\mathrm{d}}{\mathrm{d}t}(x_2-x_3) = a\left(\frac{(x_2-x_3)\cdot(x_2-x_1)^\perp}{C^{\alpha+1}} - \frac{(x_2-x_3)\cdot(x_3-x_1)^\perp}{B^{\alpha+1}}\right).$$

In the case of the 3 vortex problem [2, 72], the equations above can be rewritten using $\triangle$, the area of the direct triangle (x_1, x_2, x_3). We have that

$$(x_2-x_3)\cdot(x_2-x_1)^\perp = (x_3-x_1)\cdot(x_3-x_2)^\perp = (x_1-x_2)\cdot(x_1-x_3)^\perp = -2\triangle.$$

This gives :

$$\frac{\mathrm{d}A^2}{\mathrm{d}t} = 4a\triangle\left(\frac{1}{B^{\alpha+1}} - \frac{1}{C^{\alpha+1}}\right).$$

Similar computations lead to

$$\frac{\mathrm{d}B^2}{\mathrm{d}t} = 4\triangle\left(\frac{1}{C^{\alpha+1}} - \frac{1}{A^{\alpha+1}}\right)$$

and

$$\frac{\mathrm{d}C^2}{\mathrm{d}t} = 4\Delta\left(\frac{1}{A^{\alpha+1}} - \frac{1}{B^{\alpha+1}}\right).$$

If we define $X := (A^2, B^2, C^2)$, then $t \mapsto X(t)$ is a solution of an autonomous differential equation :

$$\frac{\mathrm{d}X}{\mathrm{d}t} = F(X),$$

where $F : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by the 3 previous equations.

We now introduce another differential system with unknown $\widetilde{X} = (\widetilde{A}^2, \widetilde{B}^2, \widetilde{C}^2)$ defined by :

$$\frac{\mathrm{d}\widetilde{X}}{\mathrm{d}t} = 4a\widetilde{\Delta}\left(\frac{1}{\widetilde{B}^{\alpha+1}} - \frac{1}{\widetilde{C}^{\alpha+1}}\right)\begin{pmatrix}1\\ \lambda^2\\ 1+\lambda^2\end{pmatrix},$$

where $\widetilde{\Delta}$ is the area of a triangle which sides are $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$. We consider the same initial datum $\widetilde{X}(0) = X(0)$. By construction we have that $\widetilde{B} = \lambda\widetilde{A}$ and $\widetilde{C} = \sqrt{1+\lambda^2}\widetilde{A}$ at every time. Indeed, these two relations hold at time 0 and hold for the the derivatives in time for all times.

The aim is to prove that $\widetilde{X}$ satisfies the same differential system as X for a well-chosen value of λ, so that we can deduce $X = \widetilde{X}$. We need to prove that

$$\frac{\mathrm{d}\widetilde{B}^2}{\mathrm{d}t} = 4\widetilde{\Delta}\left(\frac{1}{\widetilde{C}^{\alpha+1}} - \frac{1}{\widetilde{A}^{\alpha+1}}\right), \tag{5.4.10}$$

and

$$\frac{\mathrm{d}\widetilde{C}^2}{\mathrm{d}t} = 4\widetilde{\Delta}\left(\frac{1}{\widetilde{A}^{\alpha+1}} - \frac{1}{\widetilde{B}^{\alpha+1}}\right). \tag{5.4.11}$$

Expressing $\frac{\mathrm{d}\widetilde{B}^2}{\mathrm{d}t}$, $\widetilde{B}$ and $\widetilde{C}$ in terms of $\widetilde{A}$ in relation (5.4.10) gives that (5.4.10) is equivalent to

$$4\lambda^2 a\widetilde{\Delta}\left(\frac{1}{(\lambda\widetilde{A})^{\alpha+1}} - \frac{1}{(\sqrt{1+\lambda^2}\widetilde{A})^{\alpha+1}}\right) = 4\widetilde{\Delta}\left(\frac{1}{(\sqrt{1+\lambda^2}\widetilde{A})^{\alpha+1}} - \frac{1}{\widetilde{A}^{\alpha+1}}\right).$$

Simplifying, this is equivalent until the collapse to

$$\lambda^2 a\left(\frac{1}{\lambda^{\alpha+1}} - \frac{1}{(1+\lambda^2)^{\frac{\alpha+1}{2}}}\right) = \frac{1}{(1+\lambda^2)^{\frac{\alpha+1}{2}}} - 1. \tag{5.4.12}$$

Similarly, relation (5.4.11) is equivalent to

$$a(1+\lambda^2)\left(\frac{1}{\lambda^{\alpha+1}} - \frac{1}{(1+\lambda^2)^{\frac{\alpha+1}{2}}}\right) = 1 - \frac{1}{\lambda^{\alpha+1}}. \tag{5.4.13}$$

Relations (5.4.12) and (5.4.13) form a system of two equations on a and λ :

$$\begin{cases} a &= \frac{1}{\lambda^2}\left(\frac{1}{\lambda^{\alpha+1}} - \frac{1}{(1+\lambda^2)^{\frac{\alpha+1}{2}}}\right)^{-1}\left(\frac{1}{(1+\lambda^2)^{\frac{\alpha+1}{2}}} - 1\right)\\ a &= \left(1 - \frac{1}{\lambda^{\alpha+1}}\right)(1+\lambda^2)^{-1}\left(\frac{1}{\lambda^{\alpha+1}} - \frac{1}{(1+\lambda^2)^{\frac{\alpha+1}{2}}}\right)^{-1}. \end{cases} \tag{5.4.14}$$

Consequently, we want to find a solution λ to the following equation :

$$\frac{1}{\lambda^2}\left(\frac{1}{\lambda^{\alpha+1}} - \frac{1}{(1+\lambda^2)^{\frac{\alpha+1}{2}}}\right)^{-1}\left(\frac{1}{(1+\lambda^2)^{\frac{\alpha+1}{2}}} - 1\right) = \left(1 - \frac{1}{\lambda^{\alpha+1}}\right)(1+\lambda^2)^{-1}\left(\frac{1}{\lambda^{\alpha+1}} - \frac{1}{(1+\lambda^2)^{\frac{\alpha+1}{2}}}\right)^{-1}.$$

This is equivalent to

$$\frac{1+\lambda^2}{(1+\lambda^2)^{\frac{\alpha+1}{2}}}\left(\frac{1-(1+\lambda^2)^{\frac{\alpha+1}{2}}}{\lambda^2}\right)=1-\frac{1}{\lambda^{\alpha+1}}.$$

In other words, we are looking for a root of the function g defined by

$$g(\lambda)=\frac{1+\lambda^2}{(1+\lambda^2)^{\frac{\alpha+1}{2}}}\left(\frac{1-(1+\lambda^2)^{\frac{\alpha+1}{2}}}{\lambda^2}\right)-\left(1-\frac{1}{\lambda^{\alpha+1}}\right).$$

We have for $\alpha>0$ that

$$\lim_{\lambda\to 0} g(\lambda)=+\infty.$$

On the other hand,

$$g(1)=\frac{1-2^{\frac{\alpha+1}{2}}}{2^{\frac{\alpha-1}{2}}}<0.$$

Therefore, by the intermediate value theorem applied to the continuous map g, there exists $\lambda\in(0,1)$ such that $g(\lambda)=0$. Therefore the system (5.4.14) has a solution $a\in\mathbb{R}^*_-$, $\lambda\in(0,1)$, and for these values of a and λ we have that

$$\frac{\mathrm{d}\widetilde{A}^2}{\mathrm{d}t}=4\widetilde{\Delta}\left(\frac{a}{\widetilde{B}^{\alpha+1}}-\frac{a}{\widetilde{C}^{\alpha+1}}\right),$$

$$\frac{\mathrm{d}\widetilde{B}^2}{\mathrm{d}t}=4\widetilde{\Delta}\left(\frac{1}{\widetilde{C}^{\alpha+1}}-\frac{1}{\widetilde{A}^{\alpha+1}}\right)$$

and

$$\frac{\mathrm{d}\widetilde{C}^2}{\mathrm{d}t}=4\widetilde{\Delta}\left(\frac{1}{\widetilde{A}^{\alpha+1}}-\frac{1}{\widetilde{B}^{\alpha+1}}\right),$$

namely

$$\frac{\mathrm{d}\widetilde{X}}{\mathrm{d}t}=F(\widetilde{X}).$$

Since $\widetilde{X}(0)=X(0)$, by the Cauchy Lipschitz theorem, we have that $\widetilde{X}(t)=X(t)$ for all $t\in[0,T)$. Now we prove that our solution is a collapse. By construction our configuration of point-vortices is a self-similar orthogonal triangle, so we have that $\Delta=\frac{1}{2}AB$. Therefore,

$$\frac{\mathrm{d}A^2}{\mathrm{d}t}=2a\lambda A^2\left(\frac{1}{\lambda^{\alpha+1}A^{\alpha+1}}-\frac{1}{(1+\lambda^2)^{\frac{\alpha+1}{2}}A^{\alpha+1}}\right)$$

and thus, since $a<0$, there exists a constant $K>0$ depending only on a, λ and α such that

$$\frac{\mathrm{d}A}{\mathrm{d}t}=-\frac{K}{\alpha+1}\frac{1}{A^\alpha}.$$

We integrate to get

$$A^{\alpha+1}(t)-1=-Kt,$$

and thus

$$A(t)=(1-Kt)^{\frac{1}{1+\alpha}}.$$

So there is a collapse at the time $T=\frac{1}{K}$ which depends only on a, λ and α. Since we have a collapse, we can come back to Section 5.4.1 to recall that the quantity L is conserved and must vanish (5.4.5). Applying this to our situation gives

$$a(1+\lambda^2)+a\lambda^2+1=0$$

and thus

$$a = -\frac{1}{1+2\lambda^2}.$$

This formula, somehow much simpler that the one we found, gives us directly that $a \in (-1, 0)$. Therefore, the non neutral clusters hypothesis (5.1.19) is satisfied and the center of vorticity is preserved during the motion. This concludes that our solution is a self-similar collapse according to relations (5.4.1), where we just need to translate the points so that the center of vorticity is placed at 0.

We proved the existence of a self-similar collapse. We also observed above that the trajectories are $\mathcal{C}^{0,\frac{1}{\alpha+1}}([0,T])$ and not better. So the $1/(\alpha+1)$-Hölder regularity stated in Theorem 5.1.3 is optimal. Concerning the optimality of the 1/2-Hölder regularity for the Euler point-vortex dynamics in bounded domains Ω, it is a consequence of the results established in [37].

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Annexe A

Annexe

A.1 Analyse vectorielle

A.1.1 Notations et formules élémentaires

Soit $f \in C^1(\mathbb{R}^p, \mathbb{R})$. On note ∇f le gradient de f définit par

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_p} \end{pmatrix}.$$

Soit $u \in C^1(\mathbb{R}^p, \mathbb{R}^p)$. On note $\nabla \cdot u$ la divergence de u définie par

$$\nabla \cdot u = \frac{\partial u_1}{\partial x_1} + \ldots + \frac{\partial u_p}{\partial x_p}.$$

On définit le gradient d'une fonction vectorielle comme étant

$$\nabla u = \begin{pmatrix} \nabla u_1 \\ \vdots \\ \nabla u_p \end{pmatrix}.$$

De plus, on pose

$$u \cdot \nabla u = \begin{pmatrix} u \cdot \nabla u_1 \\ \vdots \\ u \cdot \nabla u_p \end{pmatrix}.$$

Lorsque les fonctions sont définies sur un espace du type $\Omega \times \mathbb{R}_+$, on notera (x, t) leur variables, que l'on désigne naturellement par la variable d'espace et de temps. La notation ∇ sera réservée pour la dérivation selon les variables d'espace uniquement. En particulier, si $f : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ et $X : \Omega \to \Omega$ sont deux fonctions de classe C^1, alors

$$\frac{\mathrm{d}}{\mathrm{d}t} f(X(t), t) = \frac{\mathrm{d}X(t)}{\mathrm{d}t} \cdot \nabla f(X(t), t) + \frac{\partial f}{\partial t}(X(t), t).$$

On a le résultat suivant.

Proposition A.1.1 (Théorème de Green Ostrogradski)**.** *Soit $F \in C^1(\Omega)$, alors*

$$\int_\Omega \nabla \cdot F(x) \mathrm{d}x = \int_{\partial\Omega} F \cdot n \mathrm{d}s.$$

Ceci donne la formule d'intégration par partie suivante :

$$\int_\Omega F(s)\cdot\nabla g(s)\mathrm{d}x = -\int_\Omega g(s)\nabla\cdot F(s)\mathrm{d}x + \int_{\partial\Omega} g(s)F(s)\cdot n\mathrm{d}s.$$

En particulier, lorsque $\Delta f = 0$, on a

$$\int_\Omega \nabla f(x)\cdot\nabla g(x)\mathrm{d}x = \int_{\partial\Omega} g(s)\nabla f(s)\cdot n\mathrm{d}s$$

A.1.2 Théorème de Liouville.

Dans cette section, on appellera *flot* toute application $\phi : \Omega\times\mathbb{R}_+ \to \Omega$ de classe C^1 telle que pour tout $t\in\mathbb{R}_+$, l'application $x\mapsto\phi(x,t)$ soit un difféomorphisme de Ω. On note $J_\phi(x,t) = \nabla_x\phi(x,t)$ la matrice Jacobienne du flot ϕ. On a donc la propriété de changement de variable suivante.

Proposition A.1.2 (Changement de variable)**.** *Soit ϕ un flot. Alors pour toute fonction f définie sur Ω, et pour tout $t\in\mathbb{R}_+$ on a*

$$\int_\Omega f(x)\mathrm{d}x = \int_\Omega \det(J_\phi(x,t))f(\phi(x,t))\mathrm{d}x.$$

En particulier, une classe de flot qui nous intéresse est la classe des flots *incompressibles*, ou encore flot *préservant la mesure.*

Définition A.1.3 (Préservation de la mesure)**.** *Soit ϕ un flot. On dit que ϕ* préserve la mesure *si pour tout $t\in\mathbb{R}_+$ et pour tout $x\in\Omega$,* $\det(J_\phi(x,t)) = 1$.

On déduit de la formule de changement de variable que tout flot qui préserve la mesure vérifie

$$\int_\Omega f(\phi(x,0))\mathrm{d}x = \int_\Omega f(\phi(x,t))\mathrm{d}x,$$

et ce quel que soit $t\in\mathbb{R}_+$.

Du théorème de Liouville énoncé ci-après, on en déduit qu'un fluide est incompressible si et seulement si son flot est incompressible.

Théorème A.1.4 (Liouville)**.** *Soit ϕ un flot et soit u tel que $u(\phi(x,t),t) = \frac{\mathrm{d}}{\mathrm{d}t}\phi(x,t)$. Alors les propriétés suivantes sont équivalentes :*

(i) le flot préserve la mesure,

(ii) pour tout temps t, $\nabla_x\cdot u(x,t) = 0$.

Démonstration. Afin d'alléger les notations, posons $J = J_\phi$. Puisque

$$\frac{\mathrm{d}}{\mathrm{d}t}J(x,t) = \mathrm{D}_x\frac{\mathrm{d}}{\mathrm{d}t}\phi(x,t) = \mathrm{D}_x[u(\phi(x,t),t)] = \nabla_x u(\phi(x,t),t)J(x,t),$$

on a

$$J(x,t+h) = J(x,t) + h\nabla_x u(\phi(x,t),t)J(x,t) + \mathcal{O}(h^2)$$

quand $h\to 0$. En multipliant par $J^{-1}(x,t)$ et en appliquant le déterminant, on a donc

$$\det\big(J(x,t+h)J^{-1}(x,t)\big) = \det\big(I_d + h\nabla_x u(\phi(x,t),t) + \mathcal{O}(h^2)\big).$$

Or, rappelons que

$$\det M = \sum_{\sigma\in\mathfrak{S}_d}\varepsilon(\sigma)\prod_{i=1}^d M_{i,\sigma(i)},$$

où d est la dimension de l'espace dans lequel évolue x et $\mathfrak{S}_d$ est l'ensemble des permutations de $[\![1, d]\!]$. Pour toute permutation $\sigma \in \mathfrak{S}_d$ ayant $d-2$ points fixes ou moins, on a

$$\prod_{i=1}^{d} [I_d + h\nabla_x u(\phi(x,t),t) + \mathcal{O}(h^2)]_{i,\sigma(i)} = \mathcal{O}(h^2).$$

La seule permutation ayant $d-1$ points fixes ou plus étant l'identité, on obtient donc

$$\begin{aligned}\det\big(J(x,t+h)J^{-1}(x,t)\big) &= \prod_{i=1}^{d} [I_d + h\nabla_x u(\phi(x,t),t)) + \mathcal{O}(h^2)]_{i,i} + \mathcal{O}(h^2) \\ &= \prod_{i=1}^{d} (1 + h\partial_{x_i} u_i(\phi(x,t),t)) + \mathcal{O}(h^2)) \\ &= 1 + h\nabla_x \cdot u(\phi(x,t),t) + \mathcal{O}(h^2).\end{aligned}$$

Ainsi,

$$\det(J(x,t+h)) = \det(J(x,t)) + h\nabla_x \cdot u(\phi(x,t),t)\det(J(x,t)) + \mathcal{O}(h^2)$$

soit

$$\frac{\mathrm{d}}{\mathrm{d}t}\det(J(x,t)) = \det(J(x,t))\nabla_x \cdot u(\phi(x,t),t).$$

Rappelons que $\det(J(0,x)) = 1$. Ainsi, il est clair que

$$\big(\forall t \in \mathbb{R}_+,\ \det(J(x,t)) = 1\big) \iff \big(\forall (x,t) \in \Omega \times \mathbb{R}_+,\ \nabla_x \cdot u(\phi(x,t),t) = 0\big).$$

□

A.2 Fonction de Green

A.2.1 Obtenue par transformation conforme

Proposition A.2.1. *Soit Ω un domaine simplement connexe, et $T : \Omega \to D = D(0,1)$ un biholomorphisme. Alors pour tout $x \neq y \in \Omega$ on a*

$$G_\Omega(x,y) = G_D(T(x),T(y)).$$

Démonstration. Posons pour tout $x \neq y \in \Omega$, $G_\Omega(x,y) = G_D(T(x),T(y))$, et montrons que G_Ω ainsi définie est bien la fonction de Green du domaine Ω. Calculons son Laplacien au sens des distributions, en introduisant, à y fixé, $f(u) = f_y(u) = G_D(u,T(y))$, de sorte que $\Delta_x G_\Omega(x,y) = \Delta(f \circ T)(x)$. Soit $\phi \in C_c^\infty(\Omega)$ une fonction test. Puisque $f \circ T$ est une fonction localement intégrable, on a par définition

$$\langle \Delta(f\circ T)\,,\,\phi\rangle = \int_\Omega f \circ T\,\Delta\phi\,\mathrm{d}x.$$

Puisque T est un biholomorphisme, on peut effectuer le changement de variable $u = T(x)$. De plus, rappelons que T satisfait les équations de Cauchy-Riemann

$$\begin{cases}\partial_1 T_1 = \partial_2 T_2 \\ \partial_1 T_2 = -\partial_2 T_1.\end{cases}$$

Ainsi, le déterminant de la matrice Jacobienne de T est $|T'|^2$, et celui de T^{-1} est donc $|(T^{-1})'|^2$. Par conséquent, on a

$$\int_\Omega f \circ T\,\Delta\phi\,\mathrm{d}x = \int_D |(T^{-1})'|^2 f\,\Delta\phi \circ (T^{-1})\,\mathrm{d}u. \tag{A.2.1}$$

On cherche maintenant à exprimer $\Delta\phi \circ (T^{-1})$ sous forme d'un Laplacien. Pour plus de clarté, on pose $S = T^{-1}$. Calculons :

$$\begin{aligned}
\Delta\big(\phi \circ S\big) = \nabla \cdot \big(\nabla\big(\phi \circ S\big)\big) &= \nabla \cdot \begin{pmatrix} \partial_1 S_1 \partial_1 \phi \circ S + \partial_1 S_2 \partial_2 \phi \circ S \\ -\partial_1 S_2 \partial_1 \phi \circ S + \partial_1 S_1 \partial_2 \phi \circ S \end{pmatrix} \\
&= \partial_1^2 S_1 \partial_1 \phi \circ S + \partial_1 S_1 \big(\partial_1 S_1 \partial_1^2 \phi \circ S + \partial_1 S_2 \partial_1 \partial_2 \phi \circ S\big) \\
&\quad + \partial_1^2 S_2 \partial_2 \phi \circ S + \partial_1 S_2 \big(\partial_1 S_1 \partial_1 \partial_2 \phi \circ S + \partial_1 S_2 \partial_2^2 \phi \circ S\big) \\
&\quad - \partial_1 \partial_2 S_2 \partial_1 \phi \circ S - \partial_1 S_2 \big(\partial_2 S_1 \partial_1^2 \phi \circ S + \partial_2 S_2 \partial_1 \partial_2 \phi \circ S\big) \\
&\quad + \partial_1 \partial_2 S_1 \partial_2 \phi \circ S + \partial_1 S_1 \big(\partial_2 S_1 \partial_1 \partial_2 \phi \circ S + \partial_2 S_2 \partial_2^2 \phi \circ S\big) \\
&= \partial_1 \phi \circ S\big(\partial_1^2 S_1 - \partial_1 \partial_2 S_2\big) + \partial_2 \phi \circ S\big(\partial_1^2 S_2 + \partial_1 \partial_2 S_1\big) \\
&\quad + \partial_1^2 \phi \circ S\big((\partial_1 S_1)^2 + (\partial_1 S_2)^2\big) + \partial_2^2 \phi \circ S\big((\partial_1 S_1)^2 + (\partial_1 S_2)^2\big) \\
&\quad + \partial_1 \partial_2 \phi \circ S\big((2-2)\partial_1 S_1 \partial_1 S_2\big) \\
&= |S'|^2 \Delta\phi \circ S.
\end{aligned}$$

Reporter ce résultat dans la relation (A.2.1) donne :

$$\int_\Omega f \circ T \, \Delta\phi \, \mathrm{d}x = \int_D f \, \Delta(\phi \circ (T^{-1})) \, \mathrm{d}u.$$

Rappelons maintenant que $f(u) = G_D(u, T(y))$ et donc pour toute fonction ψ régulière on a

$$\int_D f(u)\Delta\psi(u)\mathrm{d}u = \int_D G_D(u, T(y))\Delta\psi(u)\mathrm{d}u = \psi(T(y))$$

et donc

$$\int_D f \, \Delta(\phi \circ (T^{-1})) \, \mathrm{d}u = \phi(y).$$

On en conclut que

$$\langle \Delta(f \circ T) \, , \, \phi \rangle = \phi(y),$$

et donc la distribution $\Delta(f \circ T)$ est la masse de Dirac δ_y. Autrement dit, $\Delta_x G_\Omega(\cdot, y) = \delta_y$, ce qui est bien l'expression attendue pour la fonction de Green du domaine Ω.

Pour conclure au fait que $G_\Omega(x, y)$ est bien la fonction de Green du domaine Ω, il reste à montrer qu'elle satisfait la condition de bord

$$\forall y \in \Omega, \qquad G_\Omega(x, y) \underset{x \to \partial\Omega}{\longrightarrow} 0.$$

Ceci est vrai puisque $G_\Omega(x, y) = G_D(T(x), T(y))$ et que si $x \to \partial\Omega$, alors $T(x) \to \partial D$, et que G_D satisfait la condition de bord

$$\forall v \in D, \qquad G_\Omega(u, v) \underset{u \to \partial D}{\longrightarrow} 0.$$

Ainsi, G_Ω est bien la fonction de Green du domaine Ω.

□

A.2.2 Circulations

Soit u donné par la loi de Biot-Savart (A.2.2), que nous rappelons ici :

$$u(x, t) = \int_\Omega \nabla_x^\perp G_\Omega(x, y)\omega(y, t)\mathrm{d}y + \sum_{j=1}^m c_j(t)\beta_j(x). \tag{A.2.2}$$

Calculons alors, à l'aide de la proposition A.1.1, pour $i \geq 1$,

$$\begin{aligned}
\xi_j &= \int_{\Gamma_j} u(s,t) \cdot (-n)^{\perp} \mathrm{d}s \\
&= \int_{\Gamma_j} \int_{\Omega} \nabla_x^{\perp} G_{\Omega}(s,y) \cdot (-n)^{\perp} \omega(y,t) \mathrm{d}y \, \mathrm{d}s + \sum_{i=1}^{m} c_i(t) \int_{\Gamma_j} \beta_i(x) \cdot (-n) \, \mathrm{d}s \\
&= -\int_{\Omega} \int_{\partial\Omega} w_j(s) \nabla_x G_{\Omega}(s,y) \cdot n \mathrm{d}s \, \omega(y,t) \mathrm{d}y + c_j(t) \\
&= -\int_{\Omega} \int_{\Omega} \nabla_x \cdot (w_j(x) \nabla_x G_{\Omega}(x,y)) \mathrm{d}x \, \omega(y,t) \mathrm{d}y + c_j(t) \\
&= -\int_{\Omega} w_j(y) \omega(y,t) \mathrm{d}y - \int_{\Omega} \int_{\Omega} \nabla w_j(x) \cdot \nabla_x G_{\omega}(x,y) \mathrm{d}x \, \mathrm{d}y + c_j(t) \\
&= -\int_{\Omega} w_j(y) \omega(y,t) \mathrm{d}y - \int_{\Omega} \int_{\partial\Omega} G_{\Omega}(s,y) \nabla w_j(s) \cdot n \mathrm{d}s \, \omega(y,t) \mathrm{d}y + c_j(t) \\
&= -\int_{\Omega} w_j(y) \omega(y,t) \mathrm{d}y + c_j(t).
\end{aligned}$$

Nous obtenons bien la formule voulue.

A.3 Quasi conservation du centre de vorticité d'un cluster

Soit $\alpha \geq 0$, $N \in \mathbb{N}^*$, $(z_i)_{1 \leq i \leq N} \in (\mathbb{R}^2)^N$, $(a_i)_{1 \leq i \leq N} \in \mathbb{R}^N$. On suppose que $(z_1, \ldots, z_N) \in \mathcal{A}$, où $\mathcal{A}$ est définit par (1.19), et que $(a_i)_{1 \leq i \leq N}$ satisfasse la condition de sous cluster non neutres (2.4). On peut donc définir B_P pour tout $P \subsetneq [\![1, N]\!]$ par la formule (5.1.20). Soit $(t \mapsto z_i(t))_{1 \leq i \leq N}$ la solution de l'α-modèle (1.27) issue de $(z_i)_{1 \leq i \leq N} \in (\mathbb{R}^2)^N$. On a alors

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} B_P(t) &= \left(\sum_{i \in P} a_i \right)^{-1} \sum_{i \in P} a_i \frac{\mathrm{d}}{\mathrm{d}t} z_i(t) \\
&= \left(\sum_{i \in P} a_i \right)^{-1} \sum_{i \in P} \sum_{\substack{1 \leq j \leq N \\ j \neq i}} a_i a_j \frac{(z_i(t) - z_j(t))^{\perp}}{|z_i(t) - z_j(t)|^{\alpha+1}} \\
&= \left(\sum_{i \in P} a_i \right)^{-1} \left(\sum_{\substack{(i,j) \in P \\ i \neq j}} a_i a_j \frac{(z_i(t) - z_j(t))^{\perp}}{|z_i(t) - z_j(t)|^{\alpha+1}} + \sum_{\substack{i \in P \\ j \notin P}} a_i a_j \frac{(z_i(t) - z_j(t))^{\perp}}{|z_i(t) - z_j(t)|^{2}} \right) \\
&= \left(\sum_{i \in P} a_i \right)^{-1} \sum_{\substack{i \in P \\ j \notin P}} a_i a_j \frac{(z_i(t) - z_j(t))^{\perp}}{|z_i(t) - z_j(t)|^{\alpha+1}},
\end{aligned}$$

par antisymétrie du terme $a_i a_j \frac{(z_i(t) - z_j(t))^{\perp}}{|z_i(t) - z_j(t)|^{\alpha+1}}$. En particulier, en posant

$$C_0 = \left| \sum_{i \in P} a_i \right|^{-1},$$

on a donc

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} B_P(t) \right| \leq \sum_{\substack{i \in P \\ j \notin P}} \frac{C_0}{|z_i(t) - z_j(t)|^{\alpha}}$$

ce qui prouve que la dynamique de l'α-modèle (1.27) satisfait bien l'hypothèse (2.8).

A.4 Images earth.nullschool.net

Le site earth.nullschool.net permet, entre autres fonctionnalités, de visualiser les courants atmosphériques et marins sur la Terre. Nous donnons les dates et paramètres employés pour obtenir les images présentées dans ce manuscrit.

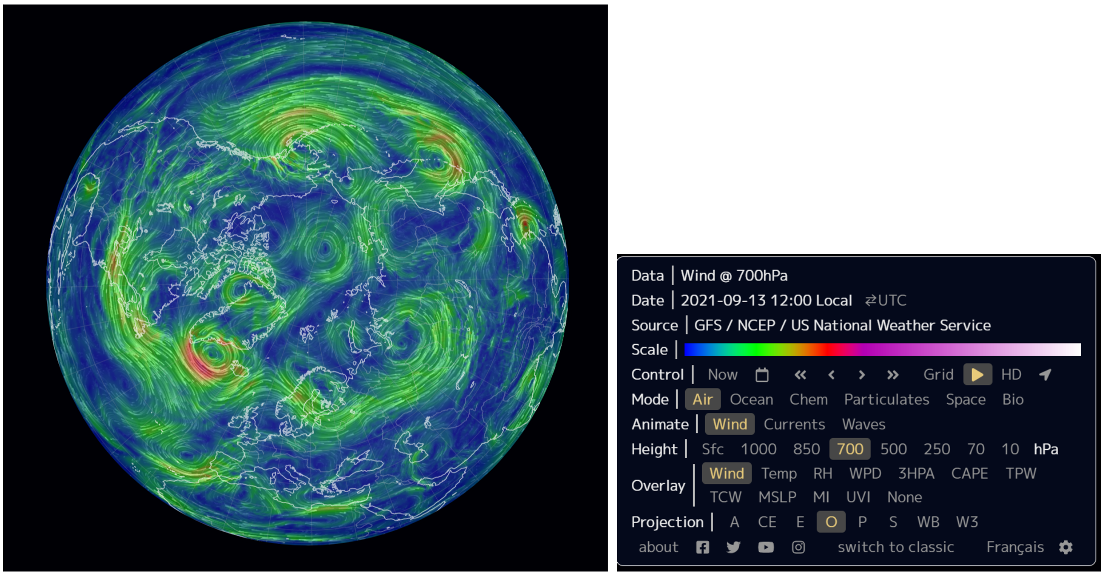

FIGURE A.1 – Paramètres de l'image donnée en Figure 1.3

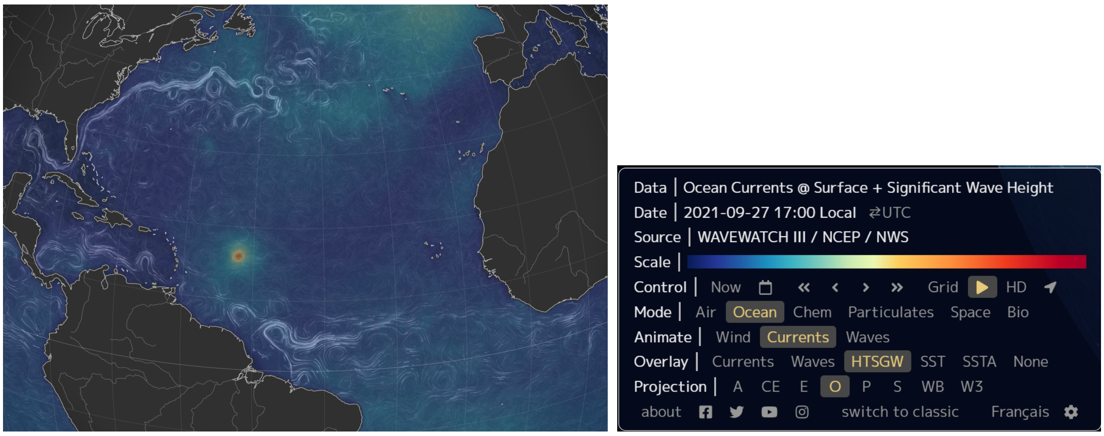

FIGURE A.2 – Paramètres de l'image donnée en Figure 1.5 à gauche

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Collisions de points-vortex et confinement dans les domaines bornés

Résumé : Dans cette thèse nous étudions les équations d'Euler 2D incompressibles dans le cas particulier d'un tourbillon très concentré autour de N points. Nous nous intéressons au système singulier limite, appelé système point-vortex. La dynamique de ce système peut produire des collisions, c'est-à-dire que la distance séparant les points-vortex peut tendre vers 0 en temps fini. Il est également possible qu'en présence d'un bord, les points-vortex collisionnent avec le bord. Dans ces deux cas, la dynamique cesse d'avoir du sens au temps de la collision.
Nous prouvons que dans les domaines bornés du plan, l'ensemble des données initiales conduisant à une collision de points-vortex est de mesure nulle. Lorsqu'une collision se produit, nous prouvons sous une hypothèse de non dégénérescence que les trajectoires des points-vortex restant loin du bord sont Hölderiennes, d'un exposant optimal. Pour les points qui collisionnent avec le bord nous prouvons que c'est leur distance au bord qui est Hölderienne. Nous prouvons également la régularité Hölderienne des trajectoires jusqu'au temps de collision dans le plan pour le système point-vortex généralisé issu des équations SQG, que nous appelons α-modèle.
Enfin, nous étudions le problème de confinement du tourbillon, dont le but est de quantifier la désingularisation : dans les équations d'Euler, si le tourbillon initial est très concentré, combien de temps reste-t-il concentré autour du système singulier ? Nous prouvons que lorsque le confinement a lieu autour d'un point bien choisi dans certains domaines bornés, le temps minimal de confinement est bien meilleur que dans le cas général.

Mots clés : Équations d'Euler incompressibles ; Système point-vortex ; Confinement du tourbillon.

Collisions of point-vortices and confinement in bounded domains

Abstract : In this thesis we study the 2D incompressible Euler equations in the case where the vorticity is sharply concentrated around N points. We are interested into the singular system obtained in the limit, called the point-vortex system. This dynamics can lead to collapses, namely the distance of point-vortices can go to 0 in finite time. It is also possible that in presence of a boundary, point-vortices hit the boundary. In both cases, the dynamics makes no sense from the time of collision.
We prove that in bounded domains, the initial data leading to a collapse are exceptional. When a collision happens, we prove under a non degeneracy hypothesis that the trajectory of the point-vortices staying far from the boundary is Hölder regular, with an optimal exponent. For the points that go to the boundary it is their distance to the boundary that is Hölder regular. We prove that the Hölder regularity of the trajectories also stands true for the generalized point-vortex system coming from the SQG equations, that we call α-model.
Finally, we study the vorticity confinement problem, which goal is to quantify the desingularization : if the vorticity is sharply concentrated, how long does it stay concentrated around the singular system ? We prove that when the confinement happens around a well chosen point in some bounded domains, the lower bound of the time of confinement is much better that in the general case.

Keywords : 2D incompressible Euler equations ; Point-vortex system ; Vorticity confinement.

Image en couverture : *Mécanique poétique des fluides*, avec l'autorisation de l'artiste, Alain Rapin.