On inviscid damping in Euler’s 2d equations

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Abstract
This paper is based on [GV14], which is a review for the Séminaire Bourbaki of [BM13]. The subject is the mathematical stability of the Couette shearing profile in the context of the two dimensional Euler’s equation for an inviscid and incompressible fluid in the class of Gevrey functions.

1 Introduction
In the following, we will consider the two dimensional Euler’s equations in order to model an incompressible flow of a fluid:

\[
\begin{cases}
\partial_t u + u \cdot \nabla u = -\nabla p \\
\nabla \cdot u = 0,
\end{cases}
\]  

(1)

where \( u \) is the speed vector field defined on a subset of \([0, \infty[ \times \mathbb{R}^2\), and \( p \) denotes the pressure inside the fluid, which is in this model the only force applying on the fluid.

Let’s consider water running freely through a pipe. The speed distribution will not be homogeneous, but instead it will tend to 0 near the boundary, while being maximum at the centre of the pipe. This kind of speed distribution is called shearing profile, because fluid particles are not moving as a wave front.

In this paper, we will consider a particular shearing profile called the Couette profile, in reference to Maurice Couette (1858-1943), a French physicist. This profile can be theoretically obtained by placing the fluid between two moving planes. The speed will then be linearly distributed in order to match both boundaries speed. This leads us to consider a special solution of equations (1):

\[ u_c(t, x, y) = (y, 0). \]  

(2)

One natural question that arises from this kind of problem is the stability of the solution. In this context, one has to keep in mind that we are working with an inviscid fluid, while the physical description of the solution was obtained thanks to the viscosity of real-life fluids only. For the same reason, a stability result would not be especially awaited as the inviscid fluid isn’t capable of dissipating energy. Therefore, to find an equilibrium may be surprising.

However, the main result of this paper, a Jacob Bedrossian and Nader Masmoudi theorem [BM13], states that in a certain sense, we have the stability of a family of shearing profiles. This result is not without recalling the Fields Medal awarded Mouhot-Villani theorem [MV11] about Landau damping for the Vlasov equation. The similarity of the two problems will be discussed later.

2 Mathematical context
In order to have an easier model to work with, we will consider now that \( y \) evolves in \( \mathbb{R} \) while \( x \) evolves in the torus \( T = \mathbb{R}/\mathbb{Z} \). This will be especially useful when working with the Fourier transform. Therefore, the Fourier variable \( k \) of \( x \) evolves in \( \mathbb{Z} \) while the Fourier variable \( \xi \) of \( y \) evolves in \( \mathbb{R} \).
Let’s introduce the vorticity, and recall some of its properties. We define \( \omega \) by
\[
\omega = \partial_x u_y - \partial_y u_x,
\]
and check easily that it satisfies the partial differential equation:
\[
\partial_t \omega + u \cdot \nabla \omega = 0.
\]
We call enstrophy the quantity \( \int \omega^2 \, dx \, dy \). Introducing \( \psi \) the function such that \( \omega = \Delta \psi \), we have that \( u = \nabla^\perp \psi \) and there comes the Biot and Savart law:
\[
u = \nabla^\perp \Delta^{-1} \omega,
\]
if the Laplacian is indeed invertible.

In the Fourier variables, this becomes:
\[
u(t, k, \xi) = \frac{1}{2\pi |x - y|^2} \int_\mathbb{R} \omega(t, x, y) e^{-ikx} e^{-i\xi y} \, dx \, dy
\]
In both cases, this means that the speed is actually completely determined by the vorticity.

Let’s compute the vorticity of the Couette flow:
\[
\omega_c(t, x, y) = -1.
\]
This means that if we take \( \Omega \) a solution of (1), \( \Omega \) its vorticity, and we define \( u = v - u_c \) the perturbation, and \( \omega = \Omega + 1 \), then we have:
\[
\partial_t \omega + y \partial_x \omega + u \cdot \nabla \omega = 0.
\]

3 Linear case

First, as a toy model, let’s consider the linear case, where in (7) we neglect completely the non linear term \( u \cdot \nabla \omega \). The equation is now:
\[
\partial_t \omega + y \partial_x \omega = 0.
\]
For this equation, we have explicit solutions. If we have some initial data \( \omega_0 \), then the solution is
\[
\omega(t, x, y) = \omega_0(x - ty, y).
\]
Let’s check what this equality becomes in the Fourier variables:
\[
\hat{\omega}(t, k, \xi) = \frac{1}{2\pi} \int_\mathbb{R} \omega(t, x, y) e^{-ikx} e^{-i\xi y} \, dx \, dy = \frac{1}{2\pi} \int_\mathbb{R} \omega_0(x, y) e^{-ik(x+yt)} e^{-i\xi y} \, dx \, dy = \hat{\omega}_0(k, \xi + kt).
\]
Now take any \( k \in \mathbb{Z} \) which is not 0, then \( |\xi + kt| \xrightarrow{t \to \infty} +\infty \) for any \( \xi \in \mathbb{R} \). This means that even if the enstrophy is preserved\(^1\), it is transported to infinity for \( x \) non 0 modes. In particular, by putting together this and the Biot and Savart law (5), we now have the estimates:
\[
\|P_{\neq 0} u_x\|_{L^2} \leq C \frac{1}{1 + t} \|\omega_0\|_{H^2}, \quad \|P_{\neq 0} u_y\|_{L^2} \leq C \frac{1}{1 + t^2} \|\omega_0\|_{H^2},
\]
\(^1\)Easy to check by change of variables.
where $P_{\neq 0}$ is the projector on the non zero Fourier modes on $x$. In conclusion, we observe a damping (except for the mean value on $x$ for every $y$) in time of the speed. But before going to the next step, we need to understand what is the reason of this inviscid damping.

First, we said that enstrophy is transported to infinity for $k \neq 0$. Basically, this means that we have the weak $L^2$ convergence:

$$\omega(t) \rightarrow_{t \to \infty} \int_T \omega_0(x,y) \, dx.$$  \hspace{1cm} (10)

But obviously, enstrophy is preserved as it is just transported, and this alone can’t explain the damping. The second effect is the regularising effect due to the Biot and Savart law (5): the inversion of the Laplacian is well known as having this property, and of course its effect is stronger that the first order derivative $\nabla^\perp$ which is applied after. It’s even clearer in the Fourier variables: we multiply by $k$ and $\xi$ and divide by $|k|^2 + |\xi|^2$. Therefore, high frequencies are decreasing because of this regularisation effect. But we just said that all the enstrophy was transported to high frequencies. Thereby the weak convergence of $\omega$ leads to a strong $L^2$ convergence for the speed $u$:

$$u(t) \rightarrow_{t \to \infty} \int_T u_0(x,y) \, dx.$$  \hspace{1cm} (11)

### 4 Main result

**Theorem 1** (Bedrossian, Masmoudi - 2013). Under some assumptions on $\omega_0$, one of them being $\|\omega_0\|_{G^{\lambda,s}}$ small enough for some $1/2 < s \leq 1$ and $\lambda_0 > 0$, then there exists some function $u_\infty$ of $y$ such that:

$$v(t) \rightarrow_{t \to \infty} (u_\infty + y, 0),$$  \hspace{1cm} (12)

with same convergence speed than in the linear case.

The full statement is much longer and very subtle. Our goal here isn’t to go into every details of the theorem or its proof, but to explain briefly its design. First, we have to introduce the Gevrey norm:

$$\|f\|_{G^{\lambda,s}} = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} e^{2\lambda(|k| + |\eta|)^s} |\hat{f}(k,\eta)|^2 \, d\eta.$$  \hspace{1cm} (13)

This norm is clearly much stronger than any $H^s$ norm for example, so an explanation is needed to its use. Let’s get back to the equation (7). As we saw, the variable $z = x - ty$ seems to play a major role in the study of the equation. Let’s see what (7) becomes changing $\omega(x,y)$ for $f(x,z)$, and $u(x,y)$ for $\bar{u}(x,z)$:

$$\partial_t f + \bar{u} \cdot \nabla f = 0.$$  \hspace{1cm} (14)

So basically, what changed is we got rid of the transport term, to reduce the equation to its non linear term, and the time derivative. If we reproduce the calculus from the linear approximation with this new form, (or make the change of variables directly after our previous results) and if we note $\eta$ the Fourier variable of $z$, we have:

$$\hat{\bar{u}}(t,k\eta) = \frac{\hat{f}_0(k\eta)}{k^2 + (\eta - kt)^2} \begin{pmatrix} i\eta \\ -ik \end{pmatrix}.$$  \hspace{1cm} (15)

This equation shows something very important: even if every non 0 mode in $x$ is decreasing in long time, they actually first increase until the resonating time $t = \eta/k$.

These resonances at various times $\eta/k$ for $k \in \mathbb{Z}$, $\eta \in \mathbb{R}$, and $k \neq 0$, are very difficult to deal with. This is the reason of the use of the Gevrey norm. Indeed, the global effect of resonances can be shown to be of order $e^{c\sqrt{\eta}}$, and when $s > 1/2$, the Gevrey space $G^{\lambda,s}$ is sent by multiplying by $e^{c\sqrt{\eta}}$ into any

\footnote{I bring to M.Sikorav’s attention that *resonating* is indeed the right word to use here.}
Gevrey space $G^{\lambda,s}$ with $\lambda' < \lambda$. In conclusion, if we control the $G^{\lambda,s}$ norm of the initial data, the resonances should not be an issue to control the $G^{\lambda',s}$ norm of the solution for every time.

The proof of the theorem also use paraproduct, which we will define now. First we decompose functions onto a base $\psi_N$ of functions on the space of frequencies $(k,\eta)$, where $N \in \{1/2, 1, 2, 4, 8, \ldots\}$, and $\sum_N \psi_N = 1$, and $\psi_N$ is supported in $D(0,1)$ for $N = 1/2$, else in the crown $C(0,N/2,3N/2)$. That way, we can write:

$$F = \sum_N \psi_N F = \sum_N F_N.$$  \hspace{1cm} (16)

Let’s write $F < k = \sum_{N<k} F_N$. Then one can write:

$$FG = \sum_N F_N G_{<N/8} + \sum_N F_{<N/8} G_N + \sum_N \sum_{N/8 < N' < 8N} F_N G_{N'}$$  \hspace{1cm} (17)

$$= T_G F + T_F G + R(F,G).$$ \hspace{1cm} (18)

The quantity $T_G F$ is called paraproduct of $G$ by $F$. Essentially, this is the product between the low frequencies of $G$ by higher frequencies of $F$ (summed over all frequencies of $F$). The term $R(F,G)$ is a rest. The interest of the paraproduct in the proof, is that this decomposition is well adapted to the efficient control of the Gevrey norm of the solution. The paraproduct satisfies good looking inequalities related to this norm and well behaving under Fourier transform.

5 Discussion

There are many things to discuss about this result. First thing first, one has to realise that the actual experiment of the Couette flow showed the instability of this shearing profile in reality. The reason of this instability is still to be understood, but we can start by criticising the model we used.

There are three major hypotheses that can induce a big difference to reality: the use of the two-dimensional model (instead of three), the use of inviscid fluid (instead of viscous), and the use of $\mathbb{T} \times \mathbb{R}$ instead of $\mathbb{R} \times \mathbb{R}$ or bounded subsets of it with boundary conditions. Even if the first point is not that bad, the two others are really strong hypotheses that may not be accurate to describe reality, especially the definition domain issue.

However, this result is still a very important mathematical result. In particular, the result is pretty similar to what Cédric Villani and Clément Mouhot obtained for the Vlasov equation in [MV11], that describes the dynamic of a plasma. They proved spontaneous convergence towards equilibrium for non-dissipative plasma, which was conjectured by Landau half a century before. This Bedrossian - Masmoudi theorem [BM13] is in the continuity of this work. It’s worth noticing that the introduction of paraproduct in this proof have inspired Clément Mouhot [BMM13] too, who has been able to help to write again the proof of Landau Damping in a shorter way thanks to these tools.

References


